Homotopy types of one-dimensional Peano continua

Katsuya Eda

Department of Mathematics, Waseda University, Tokyo 169-8555, JAPAN

2010 July

Results

Theorem 1. **Let X and Y be one-dimensional Peano continua. If the fundamental groups of X and Y are isomorphic, then X and Y are homotopy equivalent.**

Results

Theorem 1. Let X and Y be one-dimensional Peano continua. If the fundamental groups of X and Y are isomorphic, then X and Y are homotopy equivalent.

Theorem 2. Let X be a one-dimensional Peano continuum, Y a one-dimensional metric space and $x \in X$ and $y \in Y$. For each homomorphism $h : \pi_1(X, x) \to \pi_1(Y, y)$ there exists a continuous map $f : X \to Y$ and a path q from $f(x)$ to y such that $h = \varphi_q \circ f_*$, where φ_q is the base point change isomorphism.

Results continued and our strategy

Corollay 3. Let X and Y be one-dimensional Peano continua and $f : X \to Y$ a continuous map. If f induces an isomorphism between the fundamental groups of X and Y, f is a homotopy equivalence between X and Y.

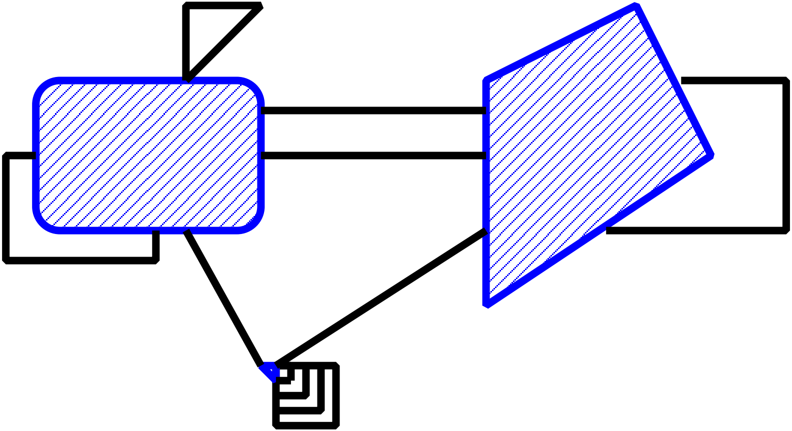
Our strategy of proofs:

Reduce a one-dimensional Peano continuum to a one-dimensional Peano continuum which is a disjoint union of wild points and open arcs. Wild points are mapped continuously by a given homomorphism according to a previous result. Hence devise a mapping on open arcs.

Reduction of one-dimensional Peano continua

X^w is the set of all wild points x, i.e. X is not semi-locally simply connected at x. O^X is the complement of X^w.

Theorem 4. (M. Meilstrup [MM]) Every one-dimensional Peano continuum is homotopy equivalent to a one-dimensional Peano continuum X such that X is a finite connected graph or O^X is at most countable union of open arcs the end points of which belong to X^w.



Main Lemma 1 (Lemma 5.1 of [E])

X: first countable, Y: one-dimensional metric space.

$h : \pi_1(X, x) \to \pi_1(Y, y)$. Then, for $x_0 \in X_h^w$ there exists a unique point $y_0 \in Y$ such that:

For a path p in X from x_0 to x, there exists a unique path q in Y from y_0 to y up to homotopy which satisfies the following:

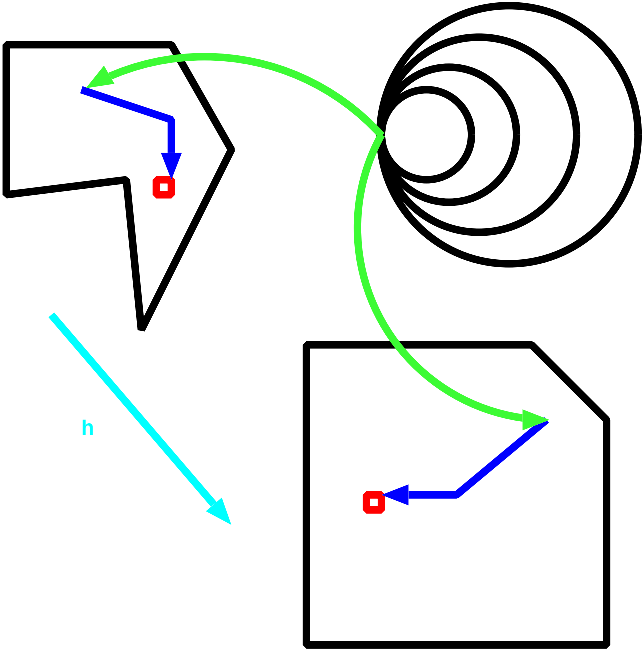
for each continuous map $f : (\mathbb{H}, o) \to (X, x_0)$ there exists a continuous map $g : (\mathbb{H}, o) \to (Y, y_0)$ such that $h \cdot \varphi_p \cdot f_* = \varphi_q \cdot g_*$.

A notion in Main Lemma 1

A point x_0 is wild with respect to h, if each neighborhood of x_0 contains a loop f with base point x_0 such that $h(\varphi_g([f]))$ is non-trivial for some path g from x_0 to x (This does not depend on the choice of g).

Define X_h^w to be the set of all points which are wild with respect to h.

If h is the injective homomorphism, $X_h^w = X^w$.


h

Map $\tilde{h}$ by Main Lemma 1

Define $\tilde{h} : X_h^w \to Y^w$ by

$$\tilde{h}(x_0) = y_0.$$

If X is path-connected and locally path-connected, $\tilde{h}$ is continuous by Lemma 5.3 of [E].

We'll extend $\tilde{h}$ on open arcs.

How to map open Arcs

$P_{X_h^w, x}(X)$ is the set of all paths from points in X_h^w to x.

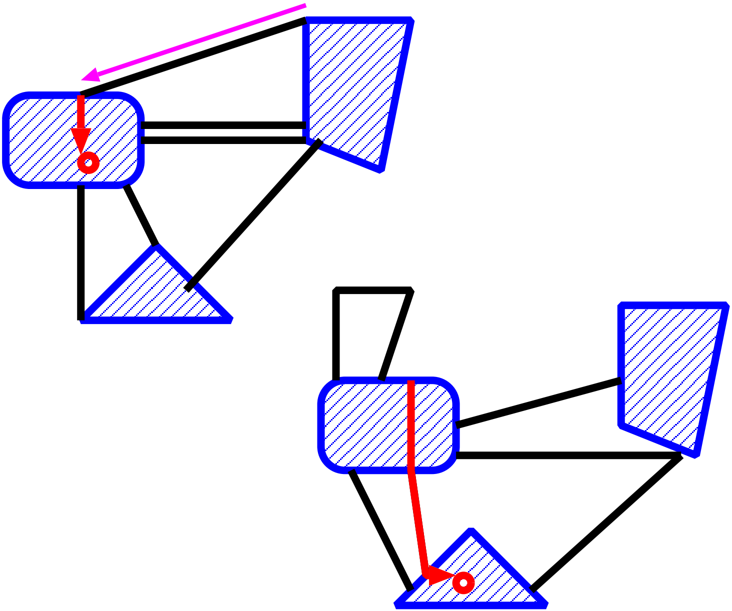
$P_{X_h^w}(X)$ is the set of all paths between points in X_h^w.

$RP(X)$ is the set of all reduced paths.

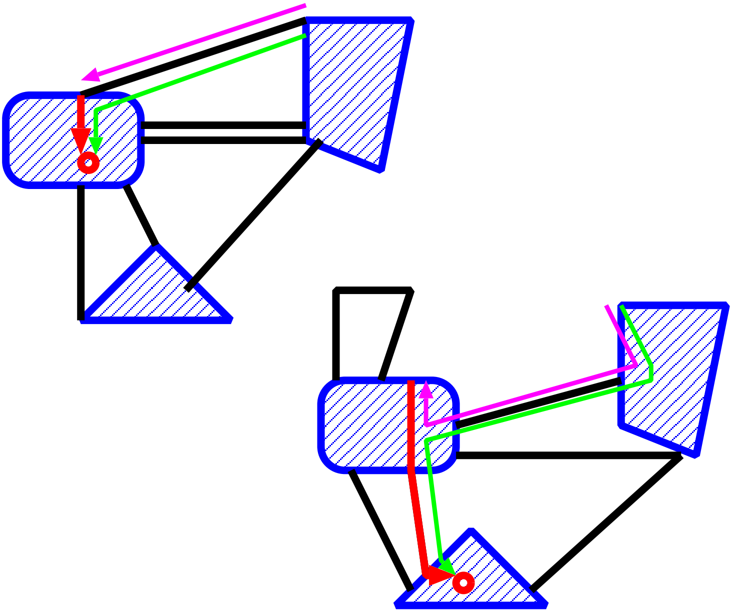
Define $\psi : P_{X_h^w, x}(X) \to RP(Y)$ by Main Lemma 1. For a path p from $x_0 \in X_h^w$ to x, we have a reduced path q from $\tilde{h}(x_0)$ to y such that the properties there hold. Let $\psi(p) = q$.

Next define $\xi : P_{X_h^w}(X) \to RP(Y)$ as follows. For a path p_0 from $x_1 \in X_h^w$ to $x_0 \in X_h^w$ in X, $p_0 p$ is a path from x_1 to x. Let $\xi(p_0)$ be a reduced path homotopic to $\psi(p_0 p)\psi(p)^-$.

How to map open Arcs (Figure 1)



How to map open Arcs (Figure 2)



Extension of $\tilde{h}$

$X = X_h^w \cup \bigcup_{i \in I} A_i$, where A_is are open arcs and $A_i^0, A_i^1 \in X_h^w$.

For each A_i, choose a continuous map $a_i : [0,1] \to \overline{A_i}$ with $a_i(0) = A_i^0$ and $a_i(1) = A_i^1$ so that the restriction of a_i to $(0,1)$ is injective. (That is, a_i is a homeomorphism, if $A_i^0 \neq A_i^1$.)

For $x \in A_i$, define

$$\tilde{h}(x) = \xi(a_i)(a_i^{-1}(x))$$

Main Lemma 2 (Lemma 6.6 of [E])

Let X be a one-dimensional metric space, $P^h_x(X)$ be the space of paths to x in the homotopy category fixing the end points and $F : [0, 1] \longrightarrow P^h_x(X)$ be a path such that $F(0)$ is degenerate. If $f \in RP_x(X)$ represents $F(1)$, then $\sigma \circ F$ and f are homotopic.

Here, $\sigma(g)$ is the starting point of a path $g \in P_x(X)$, i.e. g is a path from $\sigma(g)$ to x.

What need to be shown

(1) The definition of $\xi(p_0)$ does not depend on p. More precisely $\xi(p_0)$ is defined by p_0 and h uniquely up to the equivalence.

(2) Let $x_0, x_1, x_2 \in X_h^w$ and p_0 be a path from x_1 to x_0 and p_1 be a path from x_2 to x_1. Then $[\xi(p_1 p_0)] = [\xi(p_1)\xi(p_0)]$.

(3) For a path p_0 from $x_1 \in X_{h\circ g}^w$ to $x_0 \in X_{h\circ g}^w$, $[\xi_1(\xi_0(p_0))] = [\xi_2(p_0)]$.

What need to be shown (continued)

(4) **Let X, Y be one-dimensional metric spaces and X be locally path-connected and path-connected, and $h : \pi_1(X, x) \to \pi_1(Y, y)$ be a homomorphism. Let $x_n, y_n \in X_h^w$ and p_n be a path from y_n to x_n for each $n < \omega$ such that $\mathrm{Im}(p_n)$ converge to $x_\infty \in X_h^w$. Then $\mathrm{Im}(\xi(p_n))$ converge to $\tilde{h}(x_\infty)$.**

What need to be shown (continued again)

(5) **Let $h : \pi_1(X, x) \longrightarrow \pi_1(Y, y)$ be a homomorphism and r be a reduced path from $x_1 \in X_h^w$ to $x_0 \in X_h^w$. Then $\xi(r)$ is homotopic to $\tilde{h} \circ r$.**

(6) **Let $h_0 : \pi_1(X, x) \longrightarrow \pi_1(Y, y)$ be an isomorphism and h_1 be its inverse. Let p be a path between points in X^w. Then p is homotopic to $\tilde{h_1} \circ \tilde{h_0} \circ p$.**

Proof of Theorem 1

Assume that $X = X^w \cup \bigcup_{i \in I} A_i$ and $Y = Y^w \cup \bigcup_{j \in J} B_j$, where I and J are at most countable, A_i and B_j are open arcs. Let $h_0 : \pi_1(X, x) \to \pi_1(Y, y)$ be an isomorphism and $h_1 : \pi_1(Y, y) \to \pi_1(X, x)$ be its inverse.

By (6) a_i and $\tilde{h_1} \circ \tilde{h_0} \circ a_i$ is homotopic for each i. By (4) and a well-known fact we can join these homotopies as one homotopy.

Proof of Theorem 2

What we need to overcome is the situation "there are wild parts in X where are not wild respect to h."

Need to construct graphs in $X \setminus X_h^w$ by collapsing the wildness in X^w. Modifying proof of Theorem 4 we construct a Peano continuum which is a union of X_h^w and open arcs. Construct a retraction using that a finite graph is an absolute extensor. For this purpose brick partitions work well, since components are disjoint.

Conclusion - Reflexivity

Specker's theorem implies that the countable direct product $\mathbb{Z}^\omega$ is $\mathbb{Z}$-reflexive.

Through the Higman theorem, i.e. the non-commutative Specker theorem, this reflexivity supports these correspondences between the categories of the homotopy types and of the fundamental groups in the world of one dimensional Peano continua.

References

[E] K. Eda, The fundamental groups of one-dimensional spaces and spatial homomorphisms, Topology Appl. 123 (2002), 479–505.

[MM] M. Meilstrup, Classifying homotopy types of one-dimensional Peano continua, 2005, Master Thesis, Brigham Young University.

[MOT] J.C.Mayer, Lex G.Oversteengen, and E.D.Tymchatyn, The Menger curve characterization and extension of homeomorphisms of non-locally-separating closed subsets, PWN Dissert. Math. CCLII, 1986.