Atomic property of the fundamental groups of the Hawaiian earring and wild locally path-connected spaces

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What is atomic property?

A group has this property, if it is difficult to split it to a free product essentially.

There are many groups which are not free products of their subgroups. On the other hand, if a space has a cut point and the point has a simply- connected neighborhood the fundamental group is a free product of the fundamental groups of the two spaces separated by the point.

The fundamental group of the Hawaiian earring is a non-trivial free product of their subgroups, but it is not a free product of their uncountable subgroups.

Theorem 1. (E1(2010)) Let G_i $(i \in I)$ and H_j $(j \in J)$ be groups and $h : \times^{\sigma}_{i \in I} G_i \to *_{j \in J} H_j$ be a homomorphism from the free σ-product of groups G_i to the free product of groups H_j. Then there exist a finite subset F of I and $j \in J$ such that $h(\times^{\sigma}_{i \in I \setminus F} G_i)$ is contained in a subgroup which is conjugate to H_j.

In a version of E2(1992) the conclusion was:

$h(\times^{\sigma}_{i \in I \setminus F} G_i)$ is contained in $*_{j \in J_0} H_j$ for some finite $J_0 \subseteq J$.

Corollary 2. Let $h : \pi_1(\mathbb{H}) \to *_{j \in J} H_j$ be a homomorphism and $\mathbb{H}_n$ be the Hawaiian earing consisting of all the circles numbered by numbers greater than n. Then there exist an n such that $h(\mathbb{H}_n)$ is contained in a subgroup which is conjugate to H_j.

That is, a large part of the image is essentially contained in a factor.

Theorem 3. [E1](2010) Let X be a path-connected, locally path-connected, first countable space which is not semi-locally simply connected at any point and $h : \pi_1(X, x_0) \to *_{j \in J} H_j$ be an injective homomorphism. Then the image of h is contained in a conjugate subgroup to some H_j.

M. Bridson [B](1999) constructed a finitely presented group Γ which satisfies a polynomial isoperimetric inequality, but the asymptotic cone $\mathrm{Cone}_{\mathcal{U}}\, \Gamma$ is not simply connected. Bridson mentioned that the space is not semi-locally simply connected at any point. Hence its fundamental group cannot be decomposed to nontrivial free products by Thm 3.

Main Lemma for words

[E1,E2] Let H_j $(j \in J)$ be groups and $m, n, k \in \mathbb{N}$ such that $m + n + 2 \leq k$. Also let $y_i, z \in *_{j\in J} H_j$ $(1 \leq i \leq M)$ be elements of the free product of H_j. If the element $u = y_1 z^k \cdots y_M z^k$ satisfies $l(u) \leq m$ and $l(y_i) \leq n$ for all $1 \leq i \leq M$, then one of the following holds:

(1) z is a conjugate to an element of some H_j;

(2) there exist $j, j' \in J$, $i \in \{2, \cdots, M\}$, $f \in H_j$, $g \in H_{j'}$, $x, y \in *_{j\in J} H_j$ and a non-negative integer r such that $f^2 = g^2 = e$, $z = x^{-1} f x y^{-1} g y$, and $y_i = z^r x^{-1} f x$ or $y_i = y^{-1} g y z^r$.

References

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