Covering maps over topological groups (joint work with V. Matijević)

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Covering maps

A continuous map $f : X \to Y$ is a covering map, if for each $y \in Y$ there exists a neighborhood U of y such that $f^{-1}(U)$ is a discrete family of copies of U.
Each copy of U is called a sheet.
Infinite-sheeted covering:

$$f : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$$

Finite-sheeted covering:

$$f_p : \mathbb{R}/p\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

Covering maps over topological groups

Let $f : X \to Y$ be a covering map from a connected spaces over a toplological group Y (Y is automatically connected).

(Old question) Does there exist a group-strucure on X such that X is a topological group and f is a homomorphism?

YES means that there exists such a group-structure on X.

Well-known fact: (1) If Y is locally path-connected, then YES (in Pontrjagin's book [P] of 1950).
Less-known fact: If Y is compact and f is finite-sheeted, then YES (after 2000 by S. A. Grigorov - R. N. Gumerov [GG] and by K. E. - V. Matijević [EM1] independently).

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Overlay concerns!

Let Y be compact, then YES if and only if f is an overlay map (in 2013 by K. E. - V. Matijević [EM2]).
In addition, there exists a covering map over each solenoid such that the map cannot be an overlay map.

According to the previously stated result this covering map should be infinite-sheeted. Easy to see that a homomorphism is an overlay map.

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Covering map and Overlay

In 1972 R. Fox [F] introduced the notion of an overlay which is stronger than that of a covering map.
Let $f : X \to Y$ be a continuous map and S a set of cardinality $s = \operatorname{card} S$. Then f is an s-sheeted covering map, if there exist open coverings $\mathcal{B}$ of Y and $\mathcal{A}$ of X such that

(C1) $f^{-1}(B) = \bigcup_{\sigma \in S} A_B^\sigma$ for $B \in \mathcal{B}$;

(C2) $A_B^\sigma \cap A_B^\tau = \emptyset$ for distinct $\sigma, \tau \in S$ and $B \in \mathcal{B}$;

(C3) $f|_{A_B^\sigma} : A_B^\sigma \to B$ is a homeomorphism for each $A_B^\sigma \in \mathcal{A}$.

A covering map f is an overlay, in addition if the following additional condition is fulfilled:

(C4) If $B, B' \in \mathcal{B}$ and $B \cap B' \neq \emptyset$, then every $\sigma \in S$ admits a unique $\sigma' \in S$ such that $A_B^\sigma \cap A_{B'}^{\sigma'} \neq \emptyset$.

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New question

Let $f : X \to Y$ be an overlay map from a connected space X over a topological group Y.
Can we induce a group-strucure on X so that X is a topological group and f is a homomorphism?

This general question is open!
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New results to the old question

Theorem 1. Suppose that X is f-compactly openly connected. Then, YES if and only if f is an overlay map.

X is f-compactly connected, if for each pair $x, y \in X$ there exists a connected set C such that $x, y \in C$ and $\overline{f(C)}$ is compact. The word "openly" means that C is open.

Theorem 2. Suppose that X is locally f-compactly connected, then YES.

The local f-compact connectivity is a local version of f-compact connectivity and is equivalent to the local compact connectivity, i.e.

$$\forall U (x \in U \quad \to \quad \exists V (x \in V \subseteq U \wedge \forall y \in V \exists C (x, y \in C \subseteq U \wedge C \text{ is compact and connected })))$$

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New results continued

By Theorem 1 and the Iwasawa structure theorem for locally compact connected groups we have,

Theorem 3. Let f be a covering map from a connected space X to a locally compact group Y. The following are equivalent:

(1) YES;

(2) there exists a unique group structure on X such that X is a topological group and f is a homomorphism;

(3) f is an overlay map;

The Iwasawa theorem of connected locally compact groups

Let X be a connected locally compact group. Then there exists a connected compact subgroup K and a subspace S such that $X = SK$ and S contains the identity and is homeomorphic to an euclidean space or a point. In addition, for each $x \in X$ there exist unique $s \in S$ and $k \in K$ such that $x = sk$.

By the Iwasawa theorem every connected total space over a locally compact connected group is compactly openly connected.

The total spaces constructed over solenoids are not compactly connected.

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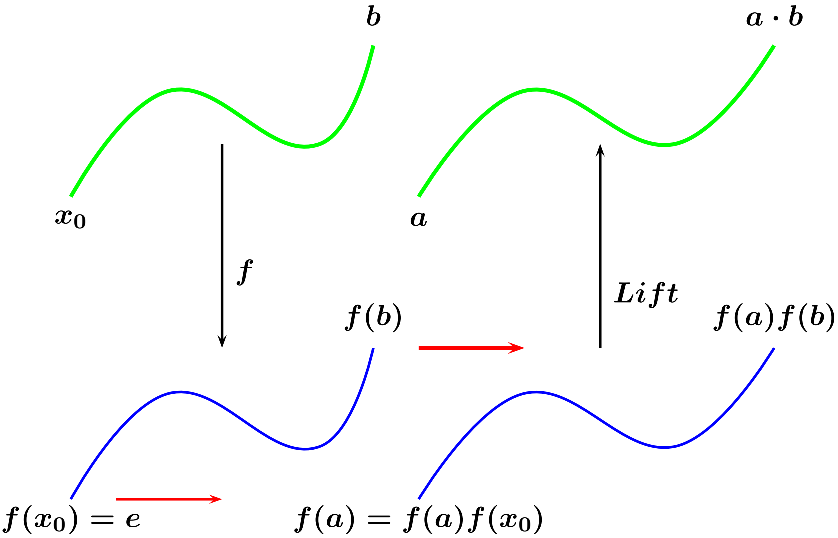
The group operation using lifting of paths



The group operation using lifting of $\mathcal{B}$-chains
Main idea of proofs of Theorems 1 and 2

Choose a suitable symmetric neighborhood V of $e \in Y$.

A sequence $a_0, \cdots, a_n \in X$ is an $\mathcal{A}$-sequence with respect to V, if it is the unique lift of the sequence $f(a_0), \cdots, f(a_n)$ where

$$f(a_{i+1}) \in f(a_i)V \cap Vf(a_i)$$

and $Vf(a_i)V \subseteq B$ for some $B \in \mathcal{B}$.

Replace paths by pointed $\mathcal{A}$-chains and $\mathcal{B}$-ones and define the operation similarly.

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Why or where does the compactness concern?

Let K be a compact subset of Y and U be a neighborhood of $e \in Y$. Then there exists a symmetric neighborhood V of $e \in Y$ such that $VyVy^{-1} \subseteq U$ for all $y \in K$.

For given a_i, b_j, suppose that $a_{i+1} \in a_iV \cap Va_i$ and $b_{j+1} \in b_jV \cap Vb_j$.
The four points $a_ib_j, a_ib_{j+1}, a_{i+1}b_j, a_{i+1}b_{j+1}$ belong to one $B \in \mathcal{B}$ and hence can be lifted to one sheet in $\mathcal{A}$.

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Key Lemmas for Proof of Theorem 3

1. Let X be a connected space and $f : X \to Y$ a covering map to a locally compact group $Y = PK$. and
Then $f^{-1}(K)$ is connected and consequently X is f-compactly openly connected.

2. Let $f : X \to Y$ be a covering homomorphism. If C is a compact subgroup of X, then the restriction $f|C : C \to f(C)$ is a covering homomorphism.

3. If two group operations $\circ$ and $\cdot$ on a locally compact group $P \circ K$ conincide on P and K respectively, then $\circ$ and $\cdot$ concide.

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Open problems

Question 1. Let $f : X \to Y$ be an overlay map from a connected space X to a topological group Y. Is there a group-structure on X such that X is a topological group and f is a homomorphism?

Question 2. Does there exist an overlay map $f : X \to Y$ from a connected space X to a topological group Y which admits two different group-structures on X such that X is a topological group and f is a homomorphism?

There are problems of uniqueness of overlay pairs up to equivalence are left. For instance, we don't know whether a covering homomorphism can be an overlay map with a non-equivalent overlay pair from the standard overlay one.

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