Infinite-sheeted coverings over solenoids (group-structures and coverings)

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Covering maps

A continuous map $f : X \to Y$ is a covering map, if for each $y \in Y$ there exists a neighborhood U of y such that $f^{-1}(U)$ is a discrete family of copies of U.

Each copy of U is called a sheet.

Infinite-sheeted covering:

$$f : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$$

Finite-sheeted covering:

$$f_p : \mathbb{R}/p\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

Covering maps over topological groups

Let $f : X \to G$ be a covering map over a toplological group G.
Can we induce a group-strucure on X so that X is a topological group and h is a homomorphism?
To avoid boring cases, we restrict ouselves to connected spaces.
Well-known fact: If G is connected and locally path-connected, then the answer is affirmative.
Less-known fact: If G is connected and compact and h is finite-sheeted, then the answer is affirmative.

Covering map and Overlay

In 1972 R. Fox introduced the notion of an overlay which is stronger than that of a covering map.
Let Y be a connected topological space, $f : X \to Y$ a continuous map, and S a set of cardinality $s = \operatorname{card} S$.
Then f is a covering map, if there exist open coverings $\mathcal{B}$ of Y and $\mathcal{A}$ of X such that

(C1) $f^{-1}(B) = \bigcup_{\sigma \in S} A_B^\sigma$ for $B \in \mathcal{B}$;

(C2) $A_B^\sigma \cap A_B^\tau = \emptyset$ for distinct $\sigma, \tau \in S$ and $B \in \mathcal{B}$;

(C3) $f|_{A_B^\sigma} : A_B^\sigma \to B$ is a homeomorphism for each $A_B^\sigma \in \mathcal{A}$.

A covering map f is an overlay, in addition if the following additional condition is fulfilled:

(C4) If $B, B' \in \mathcal{B}$ and $B \cap B' \neq \emptyset$, then every $\sigma \in S$ admits a unique $\sigma' \in S$ such that $A_B^\sigma \cap A_{B'}^{\sigma'} \neq \emptyset$.

Homomorphisms and overlays

Acoording to shape-theoretic arguments we have

Theorem 1. Let G be a connected compact group and $f : X \to G$ a covering map from a connected space. Then there exists a group-structure on X such that X is a topological group and f is a homomorphism, if and only if f is an overlay map.

According to a theorem due to Mardesić-Metijević
Corollary. Let X be a connected space, Σ a solenoid, and $f : X \to \Sigma$ an infinite-sheeted covering map over a solenoid Σ. Then X does not admit a topological group structure such that f is a covering homomorphism.

Solenoid

A solenoid Σ is a one-dimensional, connected, compact abelian group and is the Pontrjagin dual of a torsionfree abelian group of rank one, i.e. $\mathrm{Hom}^T(\Sigma, \mathbb{R}/\mathbb{Z})$ with the compact open topology.

For $A \leq \mathbb{Q}$ let $ch_A(a) = (n_0, n_1, \cdots)$ where $n_i = \max\{n : P_i^n | a \text{ in } A\}$ or $n_i = \infty$ for primes $P_0 = 2, P_1 = 3, \cdots$. The type of A is an equivalence class of $ch_A(a)$ for $a \in A$, i.e. $ch_A(a) = (n_0, n_1, \cdots) \sim ch_A(b) = (m_0, m_1, \cdots)$ if $n_k = m_k$ for almost all k and $n_k = \infty$ if and only if $m_k = \infty$. It is well-known that the isomorphism types of subgroups of $\mathbb{Q}$ is determined by types.

Presentations of a solenoid and an infinite cover

Let $\mathcal{P} = (p_0, p_1, \cdots)$ be a sequence of primes corresponding to a character, i.e. the times of the k-th prime P_k in the sequence is n_k of the character. Using p_i-sheeted covering maps over the circle group $\mathbb{R}/\mathbb{Z}$ we take the inverse limit we have a solenoid $\Sigma_{\mathcal{P}}$.
To present $\Sigma_{\mathcal{P}}$ clearly, we introduce a 0-dimensional group $J_{\mathcal{P}}$, which is a generalization of the p-adic integer group J_p.
We introduce an equivalence $\sim$ so that
$J_{\mathcal{P}} \times [0, 2\pi]/ \sim = \Sigma_{\mathcal{P}}$, i.e. $(u, 0) \sim (u - 1, 2\pi)$ for $u \in \mathbb{J}_{\mathcal{P}}$.
Let Z^i be copies of $J_{\mathcal{P}} \times [0, 2\pi]$. We define an identification $\approx$ on $\bigsqcup_{i=1}^{\infty} Z^i$ as follows.

Continued

We define $0_n, l_n \in Seq(\mathcal{P})$ as follows: $lh(0_n) = lh(l_n) = n$ and $0_{n,i} = 0$ and $l_{n,i} = p_i - 1$ for $0 \leq i < n$.
To simplify index sets, let $I_0 = 0$ and $I_n = \Sigma_{i=0}^{n-1} p_i$ for $n \geq 1$. If k is a positive integer such that $I_n + 1 \leq k \leq I_{n+1}$, then $0 \leq I_{n+1} - k \leq p_n - 1$ and $l_n * \langle I_{n+1} - k \rangle \in Seq(\mathcal{P})$. In particular, $l_0 * \langle I_1 - k \rangle = \langle p_0 - k \rangle$.

Infinite-sheeted cover

For $i \geq 1$, let Z^i be copies of $\mathbb{J}_{\mathcal{P}} \times [0, 2\pi]$ and elements of Z^i is presented by $(u, \theta)^i$ which is the copy of $(u, \theta) \in \mathbb{J}_{\mathcal{P}} \times [0, 2\pi]$.
Define $\approx$ on $\bigsqcup_{i=1}^{\infty} Z^i$, which is based on $\sim$. For $\Sigma_{i=0}^{n-1} p_i + 1 \leq k \leq \Sigma_{i=0}^{n} p_i$ and $u \in U_{l_n * \langle I_{n+1} - k \rangle}$ let $(u, 2\pi)^k \approx (u+1, 0)^{k+1}$ and $(u, 2\pi)^{k+1} \approx (u+1, 0)^k$ and in other cases let $(u, 2\pi)^k \approx (u+1, 0)^k$.

Connectivity

We denote $(U_s \times \{0\})^i$ by $i : s$ for $s \in Seq(\mathcal{P})$. Consider a fiber of intervals from $i : s$ going forward 2π. If the ends of the fibers are $i-1 : s+1$, $i : s+1$, or $i-1 : s+1$, we say it is the successor of $i : s$. Otherwise, the fibers split. Then there is no successor of $i : s$.

Suppose that $\mathcal{P}$ satisfies the property such that p_n is prime to $\Sigma_{i=0}^{n-1} p_i + 1$.

$(*_n)$ $\Sigma_{i=0}^{n-1} p_i + 2 : 0_{n+1}$ is the $(\Pi_{i=0}^{n} p_i)(\Sigma_{i=0}^{n-1} p_i + 1)$-th successor of $\Sigma_{i=0}^{n-1} p_i + 1 : 0_{n+1}$ and $k : x$ appears on that path for any $k \leq \Sigma_{i=0}^{n-1} p_i + 1$ and any $x \in Seq(\mathcal{P})$ having $lh(x) = n+1$.