# MAPS FROM THE MINIMAL GROPE TO AN ARBITRARY GROPE

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Abstract. We consider open infinite gropes and prove that every continuous map from the minimal grope to another grope is nulhomotopic unless the other grope has a 'branch' which is a copy of the minimal grope. Since every grope is the classifying space of its fundamental group, the problem is translated to group theory and a suitable block cancellation of words is used to obtain the result.

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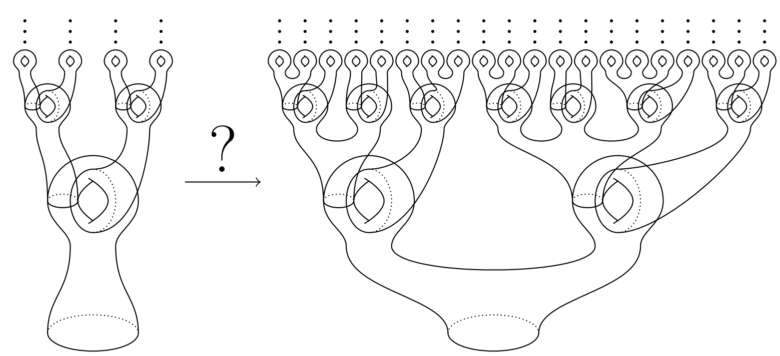
## 1. Introduction

Here we study (open infinite) gropes (a recent short note on gropes in general is [12]) and in particular we consider the question whether there exists a homotopically nontrivial map from the minimal grope to another grope.




Gropes are 2-dimensional CW complexes with infinitely many cells constructed in the following way. Start with a circle, attach a disk with handles onto the circle, onto each handle curve of the previous stage attach another disk with handles, etc. In the case of the minimal grope (also called the fundamental grope) always attach disks with only one handle.

Gropes were introduced by Štan'ko [11]. They have an important role in geometric topology ([3]; for more recent use in dimension theory see [5]

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and [4]). Their fundamental groups, which we call grope groups, were used by Berrick and Casacuberta to show that the plus-construction in algebraic K-theory is localization [2]. Recently [1] such a group has appear in the construction of a perfect group with a nonperfect localization.

Gropes are classifying spaces of their fundamental groups, so the question about the existence of homotopically nontrivial maps from the minimal grope to another grope is equivalent to the existence of nontrivial homomorphisms from the fundamental group of the minimal grope (which we call the minimal grope group) to the fundamental group of the other grope. Note that the fundamental group of the disk with one handle is the free group on two generators and that the boundary circle of this disk is homotopic to the commutator of the two free generators of the fundamental group. The disk with $n$-handles is homotopic to the one-point-union of $n$-disks with one handle and the boundary circle of the disk with $n$-handles is homotopic to the product of $n$ commutators of the free generators of the fundamental group. Thus the fundamental group of a grope is the direct limit of free groups where the connecting homomorphisms make each generator into the product of commutators of new free generators.

In algebra these groups first appeared in the proof of a lemma by Heller [8] as follows. Let $\varphi_0$ be a homomorphism from the free group $F_0$ on one generator $\alpha$ to any perfect group $P$. Let

$$\varphi_0(\alpha) = [p_0, p_1][p_2, p_3] \cdots [p_{2n-2}, p_{2n-1}] \in P. \qquad (*)$$

Then we can extend $\varphi_0$ to a homomorphism $\varphi_1$ of a (nonabelian) free group $F_1$ on $2n$ generators $\beta_0, \ldots, \beta_{2n-1}$ by setting $\varphi_1(\beta_i) = p_i$. Note that $\varphi_0(\alpha)$ may have several different expressions as a product of commutators, so we may choose any; even if some of the elements $p_1, \ldots, p_{2n-1}$ coincide, we take distinct elements $\beta_i$, $i = 1, \ldots, 2n-1$ as the generators of $F_1$. Now we repeat the above construction for every homomorphism $\varphi_1|_{\langle\beta_i\rangle}$ of the free group on one generator to $P$ and thus obtain a homomorphism $\varphi_2 : F_2 \to P$. Repeating the above construction we obtain a direct system of inclusions of free groups $F_1 \to F_2 \to F_3 \to \cdots$ and homomorphisms $\varphi_n : F_n \to P$. The direct limit of $F_n$ is a locally free perfect group $D$ and every group obtained by the above construction is called a grope group (and its classifying space is a grope). This construction shows therefore that every homomorphism from a free group on one generator to a perfect group $P$ can be extended to a homomorphism from a grope group to $P$. Note that in case the perfect group $P$ has the Ore property ([9], [6]) that every element in $P$ is a commutator, in the above process $(*)$ we can choose every generator in the chosen basis of $F_n$ to be a single commutator of two basis elements of $F_{n+1}$. The group obtained in this way is the minimal grope group $M$. The group $M$ is generated by finite nonempty words $w$ in the alphabet $\{0, 1\}$ with relations $w = [w0, w1]$, other grope groups are more difficult to define in terms of a presentation, we give a definition in Section 2. Clearly every grope group admits many epimorphisms onto $M$.

The main result of this paper is that the minimal grope admits a homotopically nontrivial map to another grope only if the other grope has a 'branch' which is another copy of the minimal grope. This seems to be one of the very few results about the existence of maps between wild spaces. Additionally, we prove that there are uncountably many self-homotopy equivalences of the minimal grope group.

In group theoretic language the main result can be formulated as follows (supporting definitions will appear in the first part of Section 2).

**Theorem 1.1.** *The minimal grope group $M = G^{S_0}$ admits a nontrivial homomorphism into a grope group $G^S$, if and only if there exists $s \in S$ such that a frame $\{t \in Seq(\mathbb{N}) : st \in S\}$ is equal to $S_0$.*

This implies, in particulary, that there exist at least two non-isomorphic grope groups (and two gropes which are not homotopically equivalent).

**Corollary 1.2.** *The minimal grope group $M = G^{S_0}$ admits a nontrivial homomorphism into a grope group $G^S$, if and only if $G^S$ is isomorphic to the free product $M * K$ for a grope group $K$.*

In Section 2 we give a systematic definition of grope groups (the combinatorics of which mimics the contruction of gropes) and prove some technical lemmas. In Section 4 we prove Theorem 1.1.

## 2. Systematic definition of grope groups and basic facts

For every positive integer $n$ let $\underline{n} = \{0, 1, \dots, n-1\}$. The set of non-negative integers is denoted by $\mathbb{N}$. We denote the set of finite sequences of elements of a set $X$ by $Seq(X)$ and the length of a sequence $s \in Seq(X)$ by $lh(s)$. The empty sequence is denoted by $\emptyset$.

For a non-empty set $A$ let $L(A)$ be the set $\{a, a^- : a \in A\}$, which we call the set of letters. We identify $(a^-)^-$ with $a$. Let $\mathcal{W}(A) = Seq(L(A))$, which we call the set of words. For a word $W \equiv a_0 \cdots a_n$, define $W^- \equiv a_n^- \cdots a_0^-$. We write $W \equiv W'$ for identity in $\mathcal{W}(A)$ while $W = W'$ for identity in the free group generated by $A$. For instance $aa^- = \emptyset$ but $aa^- \not\equiv \emptyset$. We adopt $[a, b] = aba^{-1}b^{-1}$ as the definition of the commutator. A *subword* $U$ of a word $W$ is a subsequence of $W$, i.e. $W \equiv XUY$ for some words $X$ and $Y$.

To describe all the grope groups we introduce some notation.

**Definition 2.1.** *A* grope frame $S$ *is a subset of* $Seq(\mathbb{N})$ *satisfying:*

(1) $\emptyset \in S$,
(2) *for every $s \in S$ there exists $n > 0$ such that $\underline{2n} = \{i \in \mathbb{N} : si \in S\}$,*
(3) *if the concatenation $st \in S$, for $s, t \in Seq(\mathbb{N})$, then also $s \in S$.*

In some situations it may be useful to denote the last element in (2) of the above definition by $\varepsilon(s) = 2n - 1$ where $\underline{2n} = \{i \in \mathbb{N} : si \in S\}$. If there is no ambiguity we write $\varepsilon = \varepsilon(s)$.

For each grope frame $S$ we induce formal symbols $c_s^S$ for $s \in S$ and define $E_m^S = \{c_s^S : lh(s) = m, s \in S\}$ and a free group $F_m^S = \langle E_m^S \rangle$. Then define $e^S_m : F^S_m \to F^S_{m+1}$ by $e^S_m(c^S_s) = [c^S_{s0}, c^S_{s1}] \cdots [c^S_{s\,\varepsilon(s)-1}, c^S_{s\,\varepsilon(s)}]$. Let $G^S = \varinjlim(F^S_m, e^S_m : m \in \mathbb{N})$ and $e^S_{m,n} = e^S_{n-1} \cdots e^S_m$ for $m \leq n$ and every such group $G^S$ is a *grope group*.

For $s \in S$, $s$ is *binary branched*, if $\{i \in \mathbb{N} : si \in S\} = \underline{2}$, i.e. $\varepsilon(s) = 1$. Let $S_0$ be the grope frame such that every $s \in S_0$ is binary branched, i.e. $S_0 = Seq(\underline{2})$. Then $G^{S_0} = M$ is the so-called minimal grope group. Since $e^S_m$ is injective, we frequently regard $F^S_m$ as a subgroup of $G^S$ (and similarly $F_m$ as a subgroup of $F_n$ for $m < n$).

For a non-empty word $W$ the *head* of $W$ is the left most letter $b$ of $W$, i.e. $W \equiv bX$ for some word $X$, and the *tail* of $W$ is the right most letter $c$ of $W$, i.e. $W \equiv Yc$ for some word $Y$. When $AB \equiv W$, we say that $A$ is the *head part* of $W$ and $B$ is the *tail part* of $W$. Our arguments mostly concern word theoretic arguments and we refer the reader to [7] or [10] for basic notions of words.

For a word $W \in \mathcal{W}(E^S_m)$ and $n \geq m$, we let $e^S_{m,n}[W]$ be a word in $\mathcal{W}(E^S_n)$ defined as follows: $e^S_{m,m}[W] \equiv W$ and $e^S_{m,n+1}[W]$ is obtained by replacing every $c_t$ in $e^S_{m,n}[W]$ by

$$\text{(P0)} \quad c^S_{t0}c^S_{t1}(c^S_{t0})^-(c^S_{t1})^- \cdots c^S_{t\,\varepsilon-1}c^S_{t\,\varepsilon}(c^S_{t\,\varepsilon-1})^-(c^S_{t\,\varepsilon})^-$$

and every $(c^S_t)^-$ by

$$\text{(P1)} \quad c^S_{t\,\varepsilon}c^S_{t\,\varepsilon-1}(c^S_{t\,\varepsilon})^-(c^S_{t\,\varepsilon-1})^- \cdots c^S_{t1}c^S_{t0}(c^S_{t1})^-(c^S_{t0})^-$$

respectively.

We drop the superscript $^S$, if no confusion can occur.

**Observation 2.2.** *Let $n > m+1$ and let $W \equiv e_{m+1,n}[c_{s0}]$. Suppose that $X \in \mathcal{W}(E_n)$ is a reduced word and $X \in F_m$. When $W$ is a subword of $X$, $W$ may appear in*

$$\text{(C0)} \quad e_{m,n}[c_s] = e_{m+1,n}[c_{s0}c_{s1}c^-_{s0}c^-_{s1} \cdots c_{s\varepsilon-1}c_{s\varepsilon}c^-_{s\varepsilon-1}c^-_{s\varepsilon}]$$

*or*

$$\text{(C1)} \quad e_{m,n}[c^-_s] = e_{m+1,n}[c_{s\varepsilon}c_{s\varepsilon-1}c^-_{s\varepsilon}c^-_{s\varepsilon-1} \cdots c_{s1}c_{s0}c^-_{s1}c^-_{s0}].$$

*The successive letter to $W$ in (C0) is* $\mathrm{head}(e_{m+1,n}[c_{s1}]) = c_{s10\ldots0}$*, but in (C1) it is* $\mathrm{head}(e_{m+1,n}[c^-_{s1}]) = \mathrm{head}(e_{m+2,n}[c_{s1\varepsilon(s1)}]) = c_{s1\varepsilon(s1)0\ldots0}$*. Thus the successive letter to $W$ in $X$ is not uniquely determined. However, if $X \equiv WY$ for some $Y$, the case (C1) can not appear, so the head of $Y$ is uniquely determined as $c_{s10\cdots0}$.*

*Similarly, the preceding letter to $W$ is not uniquely determined – there are four possibilities:*

- $\mathrm{tail}(e_{m+1,n}[c_{s1}]) = \mathrm{tail}(e_{m+2,n}[c^-_{s1\varepsilon(s1)}]) = \mathrm{tail}(e_{m+3,n}[c^-_{s1\varepsilon(s1)0}]) = c^-_{s1\varepsilon(s1)0\ldots0}$ *in case (C1).*
- *If $X \equiv WY$ for some $Y$, there is no preceding letter to $W$.*
- $\mathrm{tail}(e_{m,n}[c_t]) = \mathrm{tail}(e_{m+1,n}[c^-_{t\varepsilon}]) = \mathrm{tail}(e_{m+2,n}[c^-_{t\varepsilon0}]) = c^-_{t\varepsilon0\ldots0}$ *in case (C1) if $X = e_{m+1,n}[Zc_tc_sY]$ for some $Z, Y$.*

- $\mathrm{tail}(e_{m,n}[c_t^-]) = \mathrm{tail}(e_{m+1,n}[c_{t0}^-]) = c_{t0\ldots0}^-$ *in case (C1) if* $X = e_{m+1,n}[Zc_t^- c_s Y]$ *for some* $Z, Y$.

*However, the preceding letter to* $W$ *determines the succesive letter to* $W$ *uniquely:*

- *If the preceding letter to* $W$ *is* $c_{s1\varepsilon(s1)0\ldots0}^-$ *then we are in (C1).*
- *In all other cases we are in (C0).*

**Observation 2.3.** *A letter* $c_{s0\cdots0} \in \mathcal{W}(E_n)$ *for* $lh(s) = m$ *possibly appears in* $e_{m,n}[W_0]$ *in the following cases. When* $n = m+1$, $c_{s0}$ *appears once in* $e_{m,n}[c_s]$ *and also once in* $e_{m,n}[c_s^-]$. *According to the increase of* $n$, $c_{s0\cdots0}$ *appears in many parts.* $c_{s0\cdots0}$ *appears* $2^{n-m-1}$*-times in* $e_{m,n}[c_s]$ *and also* $2^{n-m-1}$*-times in* $e_{m,n}[c_s^-]$.

The following lemma is easy to verify.

**Lemma 2.4.** *For a word* $W \in \mathcal{W}(E_m)$ *and* $n \geq m$, $e_{m,n}[W]$ *is reduced, if and only if* $W$ *is reduced.*

**Lemma 2.5.** *For a reduced word* $V \in \mathcal{W}(E_n)$ *and* $n \geq m$, $V \in F_m$ *if and only if there exists* $W \in \mathcal{W}(E_m)$ *such that* $e_{m,n}[W] \equiv V$.

*Proof.* The sufficiency is obvious. To see the other direction, let $W$ be a reduced word in $\mathcal{W}(E_m)$ such that $e_{m,n}[W] = V$ in $F_n$. By Lemma 2.4 $e_{m,n}[W]$ is reduced. Since every element in $F_n$ has a unique reduced word in $\mathcal{W}(E_n)$ presenting itself, we have $e_{m,n}[W] \equiv V$. □

In the case of the minimal grope group we have

$$e_{0,2}[c_\emptyset] = c_{00}c_{01}c_{00}^-c_{01}^-c_{10}c_{11}c_{10}^-c_{11}^-c_{01}c_{00}c_{01}^-c_{00}^-c_{11}c_{10}c_{11}^-c_{10}^-,$$
$$e_{0,2}[c_\emptyset^-] = c_{10}c_{11}c_{10}^-c_{11}^-c_{00}c_{01}c_{00}^-c_{01}^-c_{11}c_{10}c_{11}^-c_{10}^-c_{01}c_{00}c_{01}^-c_{00}^-.$$

We see that the subword $c_{00}c_{01}$ (and similarly every subword of any $e_{01}[c_s^\pm]$) appears in $e_{0,2}[c_\emptyset]$ and in $e_{0,2}[c_\emptyset^-]$. On the other hand the subword $c_{01}^-c_{10}$ (and similarly every subword of $e_{02}[c_\emptyset]$ which is not a subword of $e_{01}[c_s^\pm]$) does not appears in $e_{0,2}[c_\emptyset^-]$. We generalise this observation as follows.

**Observation 2.6.** *Let us show that if a subword* $W$ *of* $e_{m,n}[d]$ *for some* $d = c_s^\pm$ *is not a subword of* $e_{m+1,n}[c_{sk}]$ *or* $e_{m+1,n}[c_{sk}^-]$, *then the word* $W$ *determines the letter* $d$ *to be either* $c_s$ *or* $c_s^-$:

*In this case* $W \equiv W_0W_1W_2$, *where* $W_0$ *might be empty while for* $i = 1, 2$ *the subword* $W_i$ *is nonempty and is the maximal subword of* $W$ *which is contained in* $e_{m+1,n}[c_{sk_i}^{\sigma_i}]$ *for some* $\sigma_i = \pm$ *and some* $k_i \in \{0, \ldots, \varepsilon(s)\}$. *We have the following four possibilities.*

1. $\sigma_1 = \sigma_2 = +$: *If* $k_2 = k_1 + 1$, *then* $k_1$ *is even and* $d = c_s$. *If, however,* $k_2 = k_1 - 1$, *then* $k_1$ *is odd and* $d = c_s^-$.
2. $\sigma_1 = +$ *and* $\sigma_2 = -$: *If* $k_2 = k_1 - 1$, *then* $k_1$ *is odd and* $d = c_s$. *If, however,* $k_2 = k_1 + 1$, *then* $k_1$ *is even and* $d = c_s^-$.

(3) $\sigma_1 = -$ *and* $\sigma_2 = +$*: If* $k_2 = k_1 + 1$*, then* $k_1$ *is odd and* $d = c_s$*. If, however,* $k_2 = k_1 - 1$*, then* $k_1$ *is even and* $d = c_s^-$*.*
(4) $\sigma_1 = \sigma_2 = -$*: If* $k_2 = k_1 + 1$*, then* $k_1$ *is even and* $d = c_s$*. If, however,* $k_2 = k_1 - 1$*, then* $k_1$ *is odd and* $d = c_s^-$*.*

*Hence the word* $W$ *determines the letter* $d$ *uniquely.*

Motivated by this observation we state the following technical definition.

**Definition 2.7.** *Let* $W_0 \in \mathcal{W}(E_m)$ *and* $n > m$*. A subword* $V \in \mathcal{W}(E_n)$ *of* $e_{m,n}[W_0]$ *is* small*, if there exists a letter* $c_s$ *or* $c_s^-$ *in* $W_0$ *and* $i \in \mathbb{N}$ *such that* $V$ *is a subword of either* $e_{m+1,n}[c_{si}]$ *or* $e_{m+1,n}[c_{si}^-]$*.*

In particular, the word $W$ in Observation 2.6 is not small. Note that being small depends on $m$. In the following usage of this notion $m$ and $n$ are always fixed in advance.

Note that a letter $c_{s0\cdots 0} \in \mathcal{W}(E_n)$ for $lh(s) = m$ possibly appears in $e_{m,n}[W_0]$ in the following cases. When $n = m + 1$, $c_{s0}$ appears once in $e_{m,n}[c_s]$ and also once in $e_{m,n}[c_s^-]$. According to the increase of $n$, $c_{s0\cdots 0}$ appears in many parts. $c_{s0\cdots 0}$ appears $2^{n-m-1}$-times in $e_{m,n}[c_s]$ and also $2^{n-m-1}$-times in $e_{m,n}[c_s^-]$. This is a particular case where a subword is small.

For an arbitrary reduced word $W \in \mathcal{W}(E_n)$ small subwords in $\mathcal{W}(E_n)$ are not defined. However, according to Lemma 2.5, if also $W \in F_m$, $m < n$, a subword of $W \in \mathcal{W}(E_n)$ is small considering $W \equiv e_{m,n}[W_0]$ for a word $W_0 \in \mathcal{W}(E_m)$.

**Lemma 2.8.** *Let* $m < n$ *and* $A$ *be a non-empty word in* $\mathcal{W}(E_n)$*. Let* $X_0AY_0$ *and* $X_1AY_1$ *be reduced words in* $\mathcal{W}(E_n)$ *satisfying* $X_0AY_0, X_1AY_1 \in F_m$*.*

(1) *If* $A$ *is not small,* $X_0A \notin F_m$ *and* $X_1A \notin F_m$*, then* $\mathrm{head}(Y_0) = \mathrm{head}(Y_1)$*.*
(2) *Let* $X_0$ *be an empty word. If* $A$ *is not small and* $A \notin F_m$*,then* $\mathrm{head}(Y_0) = \mathrm{head}(Y_1)$*.*
(3) *Let* $X_0$ *and* $X_1$ *be empty words. If* $A \notin F_m$*, then* $\mathrm{head}(Y_0) = \mathrm{head}(Y_1)$*.*

*Proof.* (1) Since $X_0AY_0 \in F_m$ but $X_0A \notin F_m$, we have a letter $c \in E_m \cup E_m^-$ and words $U_0, U_1, U_2$ such that $U_1 \not\equiv \emptyset$, $U_2 \not\equiv \emptyset$, $X_0A \equiv U_0U_1$ and $U_1U_2 \equiv e_{m,n}[c]$. Since $A$ is not small, $c$ and $U_0, U_1, U_2$ are uniquely determined by $A$. Since the same thing holds for $X_1AY_1$, we have the conclusion by Observation 2.2 for $n > m + 1$. (The case for $n = m + 1$ is easier.)

(2) Since $AY_0 \in F_m$, $A \notin F_m$ and $A$ is not a small word, for any word $B$ such that $BA$ is reduced we have $BA \notin F_m$. In particular $X_1A \notin F_m$ and the conclusion follows from (1).

(3) Since $AY_0 \in F_m$, there are $A_0$ and non-empty $U_0, U_1$ such that $A_0 \in F_m$, $A \equiv A_0U_0$ and $U_0U_1 \equiv e_{m,n}[c]$ for some $c \in E_m \cup E_m^-$. Since $A \notin F_m$, the head of $U_1$ is uniquely determined by $A$ and hence the heads of $Y_0$ and $Y_1$ are the same (Observation 2.2). □

**Lemma 2.9.** *Let $m < n$ and $A, X, Y \in \mathcal{W}(E_n)$ and $AXA^-Y \in F_m$. If $AXA^-Y$ is reduced and $A$ is not small, then $AXA^- \in F_m$ and $Y \in F_m$.*

*Proof.* The head of the reduced word in $\mathcal{W}(E_m)$ for the element $AXA^-Y$ is $c_s$ or $c_s^-$ for $c_s \in E_m$. According to $c_s$ or $c_s^-$, $A \equiv e_{m+1,n}[c_{s0}]Z$ or $e_{m+1,n}[c_{sk}]Z$ for a non-empty word $Z$, where $\underline{k+1} = \{i \in \mathbb{N} : si \in S\}$ is even. Then $A^- \equiv Z^- e_{m+1,n}[c_{s0}^-]$ or $A^- \equiv Z^- e_{m+1,n}[c_{sk}^-]$ and hence $AXA^- \in F_m$ and consequently $Y \in F_m$. □

**Lemma 2.10.** *For $e \neq x \in F_m^S$ and $u \in G^S$, $uxu^{-1} \in F_m^S$ implies $u \in F_m^S$.*

*Proof.* There exists $n \geq m$ such that $u \in F_n$. Let $W$ be a cyclically reduced word and $V$ be a reduced word such that $x = VWV^-$ in $F_m$ and $VWV^-$ is reduced. Then $e_{m,n}(x) = e_{m,n}[V]e_{m,n}[W]e_{m,n}[V]^-$ and $e_{m,n}[V]$ is reduced and $e_{m,n}[W]$ is cyclically reduced by Lemma 2.4. Let $U$ be a reduced word for $u$ in $F_n$. Let $k = lh(U)$. Then $e_{m,n}(x^{2k+1}) = e_{m,n}[V]e_{m,n}[W]^{2k+1}e_{m,n}[V]^-$ and the right hand term is a reduced word. Hence the reduced word for $ux^ku^-$ of the form $Xe_{m,n}[W]Y$, where $Ue_{m,n}[V]e_{m,n}[W]^k = X$ and $e_{m,n}[W]^k e_{m,n}[V]^- U^- = Y$. Since $ux^ku^{-1} \in F_m$, $X \in F_m$ and $Y \in F_m$. Now we have $Ue_{m,n}[V] \in e_{m,n}(F_m)$ and hence $U \in e_{m,n}(F_m)$, which implies the conclusion. □

**Lemma 2.11.** *Let $UWU^-$ be a reduced word in $\mathcal{W}(E_n)$. If $UWU^- \in F_m$ and $W$ is cyclically reduced, then $U, W \in F_m$.*

*Proof.* If $U$ is empty or $n = m$, then the conclusion is obvious. If $U \in F_m$, then $WU^- \in F_m$ and so $W \in F_m$. Suppose that $U$ is $U \notin F_m$. Since $UWU^-, UW^-U^- \in F_m$, the head of $W$ and that of $W^-$ is the same by Lemma 2.8 (3), which contradicts that $W$ is cyclically reduced. □

**Lemma 2.12.** *Let $XY$ and $YX$ be reduced words in $\mathcal{W}(E_n)$ for $n \geq m$. If $XY$ and $YX$ belong to $F_m$, then both of $X$ and $Y$ belong to $F_m$.*

*Proof.* We may assume $n > m$. When $n > m$, the head of $e_{m,n}[W]$ for a non-empty word $W \in \mathcal{W}(E_m)$ is $c_{s0\cdots0}$ or $c_{sk0\cdots0}$ where $lh(s) = m$ and $k+1 = \{i \in \mathbb{N} : si \in S\}$ is even. (When $n = m+1$, there appears no $0\cdots0$.) Since $X^-Y^- \in F_m$ and $X^-Y^-$ is reduced, the tail of $X$ is of the form $c_{s0\cdots0}^-$ or $c_{sk0\cdots0}^-$. We only deal with the former case. Suppose that $X \notin F_m$. Since $XY \in F_m$ and $XY$ is reduced, $X \equiv Ze_{m+1,n}[c_{s1}c_{s0}^-]$ for some $Z$. This implies $X^- \equiv e_{m+1,n}[c_{s0}c_{s1}^-]Z^-$, which contradicts that $X^-Y^- \in F_m$ and $X^-Y^-$ is reduced. Now we have $X, Y \in F_m$. □

**Lemma 2.13.** *Let $m < n$ and $A, B, C \in \mathcal{W}(E_n)$ such that $ABCA^-B^-C^- \in F_m$ and is nontrivial. If $ABCA^-B^-C^-$ is a reduced word and at least one of $A, B, C$ is not small, then $A, B, C \in F_m$.*

*Proof.* Since $ABCA^-B^-C^- \neq e$, at most one of $A, B, C$ is empty. When $C$ is empty, the conclusion follows from Lemma 2.9 and the fact that $BAB^-A^-$ is also reduced and $BAB^-A^- \in F_m$.

Now we assume that $A, B, C$ are non-empty. If $A$ is not small, then $ABCA^- \in F_m$ and $B^-C^- \in F_m$ by Lemma 2.9. Since $BC$ is cyclically reduced, $A \in F_m$ and $BC \in F_m$ by Lemma 2.11. The conclusion follows from Lemma 2.12. In the case that $C$ is not small, the argument is similar. The remaining case is when $A$ and $C$ are small. Then $ABCA^-B^-C^- \in F_m$ and $CBAC^-B^-A^- \in F_m$ imply $A \equiv C$, which contradicts the assumption that $ABCA^-B^-C^-$ is reduced. □

**Lemma 2.14.** *Let $m < n$ and $A, B, C \in \mathcal{W}(E_n)$ such that $ABCA^-B^-C^- \in F_m$ and is nontrivial. If $ABCA^-B^-C^-$ is a reduced word and $A, B, C$ are small, then one of $A, B, C$ is empty.*

*Assume $C$ is empty. Then there exists $c_s \in E_m$ such that $s$ is binary branched and either*

$$A \equiv e_{m+1,n}[c_{s0}] \text{ and } B \equiv e_{m+1,n}[c_{s1}],$$

*or*

$$A \equiv e_{m+1,n}[c_{s1}] \text{ and } B \equiv e_{m+1,n}[c_{s0}].$$

*Proof.* Since $A, B, C$ are small, all the words $A, B, C$ and their inverses must be subwords of $e_{m+1,n}[c_{si}]$, $i = 0, 1$, or $e_{m+1,n}[c_{si}^-]$, for an element $c_s \in E_m$, and in particular that either

$$ABCA^-B^-C^- = e_{m,n}(c_s) = e_{m+1,n}[c_{s0}c_{s1}c_{s0}^-c_{s1}^-]$$

or

$$ABCA^-B^-C^- = e_{m,n}(c_s^-) = e_{m+1,n}[c_{s1}c_{s0}c_{s1}^-c_{s0}^-],$$

where the left most and right most terms are reduced words. Note that if the cardinality of $\{i \in \mathbb{N} : si \in S\}$ were greater than 2, one of $A, B, C$ would not be small; hence in our case $s$ is binary branched. We only deal with the first case. Then $ABC \equiv e_{m+1,n}[c_{s0}c_{s1}]$ and $A^-B^-C^- \equiv e_{m+1,n}[c_{s0}^-c_{s1}^-]$. In case $A, B, C$ are non-empty, $A$ is a proper subword of $e_{m+1,n}[c_{s0}]$ or $C$ is a proper subword of $e_{m+1,n}[c_{s1}]$. In either case $A^-B^-C^- \equiv e_{m+1,n}[c_{s0}^-c_{s1}^-]$ does not hold. Hence one of $A, B, C$ is empty. We may assume $C$ is empty. Since $A, B$ are small, $A \equiv e_{m+1,n}[c_{s0}]$ and $B \equiv e_{m+1,n}[c_{s1}]$. □

## 3. Block reduction

In this section we develop the method which we use to prove Theorem 1.1. Using letter reduction Wicks [13] showed that every commutator in an arbitrary free group is cyclicaly equivalent to a word of the form $ABCA^-B^-C^-$. We generalise his appoach in order to keep track of words in $F_n$ which belong also in $F_m$, $m < n$. In particular, reducing words by certain blocks of letters we show that for words $A, B, X, Y \in \mathcal{W}(E_n)$ such that the reduced word of $Y^-ABYX^-A^-B^-X$ is cyclicaly reduced and is an element of $F_m$, then either both elements $Y^-ABY$ and $X^-A^-B^-X$ are in $F_m$ or the entire word is a generator $c_s$ of $F_m$ or its inverse $c_s^-$.

Lemmas 3.1, 3.2, 3.3 and 3.4 show connections between our reduction steps in case at least one of $X$ and $Y$ is empty. Based on the results of the previous section, these lemmas can be proved fairly easily, but they show what the block-wise reductions are. Lemma 3.5 corresponds to the final step, i.e. when we have the reduced word. Lemmas 3.6 and 3.7 correspond to the case that $X$ and $Y$ are non-empty. In lemmas of this section we assume $m < n$.

**Lemma 3.1.** *Let $A, B \in \mathcal{W}(E_n)$ be non-empty reduced words such that $ABA^-B^- \neq e$ and $AB, A^-B^-$ are reduced words. Then the following hold:*

(1.1) *If $B \equiv B_0A$, then $B_0$ is non-empty, $AB_0, A^-B_0^-$ are reduced words and the identity $AB_0A^-B_0^- = ABA^-B^-$ holds. In addition if $AB_0$, $A^-B_0^- \in F_m$, then $AB, A^-B^- \in F_m$.*

(1.2) *If $A \equiv A_0B$, then $A_0$ is non-empty, $A_0B$, $A_0^-B^-$ are reduced words and the identity $A_0BA_0^-B^- = ABA^-B^-$ holds. In addition if $A_0B$, $A_0^-B^- \in F_m$, then $AB, A^-B^- \in F_m$.*

(1.3) *If $A \equiv A_0Z$ and $B \equiv B_0Z$ for non-empty words $A_0$, $B_0$ such that $B_0A_0^-$ is reduced, then $A_0ZB_0A_0^-Z^-B_0^-$ is reduced and the identity $A_0ZB_0A_0^-Z^-B_0^- = ABA^-B^-$ holds. In addition if $A_0, B_0, Z \in F_m$, then $AB, A^-B^- \in F_m$.*

*Proof.* We only show (1.1). The non-emptiness of the word $B_0$ follows from $ABA^-B^- \neq e$. Since $AB$ and $A^-B^-$ are reduced, $AB_0$ and $A^-B_0^-$ are cyclically reduced and hence the second statement follows from Lemma 2.12. □

**Lemma 3.2.** *Let $A, B, C \in \mathcal{W}(E_n)$ be reduced words (possibly empty) such that $ABCA^-B^-C^- \neq e$ and $AB$, $CA^-B^-C^-$ are reduced words. Then the following hold:*

(2.1) *If $B \equiv B_0C^-$, then $AB_0$, $A^-CB_0^-C^-$ are reduced words and the identity $AB_0A^-CB_0^-C^- = ABCA^-B^-C^-$ holds. In addition if $AB_0A^-$, $CB_0^-C^- \in F_m$, then $AB, CA^-B^-C^- \in F_m$.*

(2.2) *If $C \equiv B^-C_0$, then $AC_0$, $A^-B^-C_0^-B$ are reduced words and the identity $AC_0A^-B^-C_0^-B = ABCA^-B^-C^-$ holds. In addition if $AC_0A^-$, $B^-C_0^-B \in F_m$, then $AB, CA^-B^-C^- \in F_m$.*

(2.3) *If $B \equiv B_0Z^-$ and $C \equiv ZC_0$ for non-empty words $B_0$, $C_0$ and $B_0C_0$ is reduced, then $AB_0C_0A^-ZB_0^-C_0^-Z^-$ is reduced and the identity $AB_0C_0A^-ZB_0^-C_0^-Z^- = ABCA^-B^-C^-$ holds. In addition if $AB_0C_0A^-$, $ZB_0^-C_0^-Z^- \in F_m$, then $AB, CA^-B^-C^- \in F_m$.*

*Proof.* (2.1) The first proposition is obvious. Let $B_0 \equiv XB_1X^-$ for a cyclically reduced word $B_1$. Since $(AX)B_1(AX)^-, (CX)B_1^-(CX)^- \in F_m$, $AX, CX, B_1 \in F_m$ by Lemma 2.11. Now $AB = (AX)B_1(CX)^- \in F_m$ and $CA^-B^-C^- = (CX)(AX)^-(CB_0^-C^-) \in F_m$. We see (2.2) similarly.

For (2.3) observe the following. Since both $B_0$ and $C_0$ are non-empty, $B_0C_0$ and $B_0^-C_0^-$ are cyclically reduced. Hence, using Lemmas 2.11 and 2.12, we have (2.3). □

The next two lemmas are stated so that they can be directly applied by pattern matching for the reduction steps and so the statements contain trivial parts.

**Lemma 3.3.** *Let $A, B, C \in \mathcal{W}(E_n)$ be reduced words (possibly empty) such that $ABA^-CB^-C^- \neq e$ and $AB$, $A^-CB^-C^-$ are reduced. Then the following hold:*

(3.1) *If $A \equiv A_0B$, then $A_0B$, $A_0^-CB^-C^-$ are reduced and the identity $A_0BA_0^-CB^-C^- = ABA^-CB^-C^-$ holds. In addition if $A_0BA_0^-$, $CB^-C^- \in F_m$, then $ABA^-, CB^-C^- \in F_m$.*

(3.2) *If $B \equiv B_0A$, then $AB_0$, $CA^-B_0^-C^-$ are reduced and the identity $AB_0CA^-B_0^-C^- = ABA^-CB^-C^-$ holds. In addition if $AB_0$, $CA^-B_0^-C^- \in F_m$, then $ABA^-, CB^-C^- \in F_m$.*

(3.3) *If $B \equiv B_0Z$ and $A \equiv A_0Z$ for non-empty words $A_0$, $B_0$ and $B_0A_0^-$ is reduced, then $A_0ZB_0A_0^-CZ^-B_0^-C^-$ is reduced. In addition if $A_0ZB_0A_0^-, CZ^-B_0^-C^- \in F_m$, then $ABA^-, CB^-C^- \in F_m$.*

*Proof.* The proof is not difficult, so we only indicate the main steps. Checking that the words are elements of $F_m$ is a matter of straightforward calculations.

(3.1) Since $A_0B$ is $A$ itself and $A_0^-CB^-C^-$ is a subword of $A^-CB^-C^-$, they are reduced by assumption.

(3.2) Since $AB_0$ is a subword of $AB$ and $CA^-B_0C^-$ is a subword of $A^-CB^-C^-$, they are reduced.

(3.3) Since $A_0ZB_0$ is a subword of $AB$ and $A_0^-CZ^-BC^-$ is a subword of $A^-CB^-C^-$, they are reduced and hence $A_0ZB_0A_0^-CZ^-B_0^-C^-$ is reduced by the assumption of (3.3). □

The following lemma can be proved similarly to the preceding Lemma 3.3 and we omit its proof.

**Lemma 3.4.** *Let $A, B, C \in \mathcal{W}(E_n)$ be reduced words (possibly empty) such that $ABA^-CB^-C^- \neq e$ and $A$, $BA^-CB^-C^-$ are reduced words. Then the following hold:*

(4.1) *If $A \equiv A_0B^-$, then $A_0$, $BA_0^-CB^-C^-$ are reduced and the identity $A_0BA_0^-CB^-C^- = ABA^-CB^-C^-$ holds. In addition if $A_0BA_0^-$, $CB^-C^- \in F_m$, then $ABA^-, CB^-C^- \in F_m$.*

(4.2) *If $B \equiv A^-B_0$, then $B_0$, $B_0A^-CB_0^-AC^-$ are reduced and the identity $B_0A^-CB_0AC^- = ABA^-CB^-C^-$ holds. In addition if $B_0A^-$, $CB_0^-AC^- \in F_m$, then $ABA^-, CB^-C^- \in F_m$.*

(4.3) *If $A \equiv A_0Z^-$ and $B \equiv ZB_0$ for non-empty words $A_0$, $B_0$ and $A_0B_0$ is reduced, then $A_0B_0ZA_0^-CB_0^-Z^-C^-$ is reduced and the identity $A_0B_0ZA_0^-CB_0^-Z^-C^- = ABA^-CB^-C^-$ holds. In addition if $A_0B_0ZA_0^-, CB_0^-Z^-C^- \in F_m$, then $ABA^-, CB^-C^- \in F_m$.*

The following lemma is used several times in the proof of the main theorem.

**Lemma 3.5.** *Let $A, B, C, D \in \mathcal{W}(E_n)$ be reduced non-empty words.*

1. *If $ABA^-B^-$ is reduced and $ABA^-B^- \in F_m$ and at least one of $A, B$ is not small, then $A, B \in F_m$.*
2. *If $ABCA^-B^-C^-$ is reduced and $ABCA^-B^-C^- \in F_m$ at least one of $A, B, C$ is not small, then $A, B, C \in F_m$.*
3. *If $CABC^-DA^-B^-D^-$ is reduced and $CABC^-DA^-B^-D^- \in F_m$, then $A, B, C, D \in F_m$.*
4. *If $CAC^-DA^-D^-$ is reduced and $CAC^-DA^-D^- \in F_m$, then $CAC^-$, $DA^-D^- \in F_m$.*

*Proof.* The statements (1) and (2) are paraphrases of Lemma 2.13.
(3) Let $c$ be the head of $C$ and $d$ be the tail of $D^-$. Since $c^-$ immediately precedes $d^-$, we have $CABC^-, DA^-B^-D^- \in F_m$. Since $AB$, $A^-B^-$ are reduced and both $A$ and $B$ are non-empty, $AB$ is cyclically reduced. Now the conclusion follows from Lemmas 2.11 and 2.12.
The proof of (4) follows the reasoning in the proof of (3). □

**Lemma 3.6.** *Let $A^-B^-$ and $X_0ABX_0^-$ be reduced words such that $X_0AB \equiv BAX_1$ for some $X_1$. If $lh(X_0) \leq lh(B)$, then there exist $A', B'$ such that $lh(B') < lh(B)$, $(A')^-(B')^-$ and $X_0A'B'X_0^-$ are reduced words, $X_0A'B' \equiv B'A'X_1$, $A^-B^-X_0ABX_0^- = (A')^-(B')^-X_0A'B'X_0^-$, and $A, B \in \langle X_0, A', B' \rangle$.*

*Proof.* First note that $lh(X_0) \neq lh(B)$ since $BX_0^-$ is reduced. Hence $lh(B) > lh(X_0)$. If $lh(B) = lh(X_0) + lh(A)$, then we have $X_0A \equiv B \equiv AX_1$ and have the conclusion, i,e, $A' \equiv A$ and $B' \equiv \emptyset$.

If $lh(B) < lh(X_0) + lh(A)$, we have $k > 0$ and $A_0, A_1$ such that $B \equiv X_0A_0A_1$, $A \equiv (A_0A_1)^kA_0$, and $A_1$ is non-empty. (Note that $A_0$ may be empty.) Let $A' \equiv A_0$ and $B' \equiv A_1$. Since $lh(X_0) + lh(A) = lh(B) + (k-1)lh(A_0A_1) + lh(A_0)$, we have $B \equiv A_1A_0X_1$. Let $A' \equiv A_0$ and $B' \equiv A_1$, then we have the conclusion.

If $lh(B) > lh(X_0) + lh(A)$, we have $k > 0$ and $B_0, B_1$ such that $B_0B_1 \equiv X_0A$, $B \equiv (B_0B_1)^kB_0$, and $B_1$ is non-empty. Note that $B_0$ may be empty.) Since $lh(B_1B_0) = lh(AX_1)$, we have $B_1B_0 \equiv AX_1$. Now $B \equiv X_0A(B_0B_1)^{k-1}B_0 \equiv (B_0B_1)^{k-1}B_0AX_1$ holds. Let $A' \equiv A$ and $B' \equiv (B_0B_1)^{k-1}$, then we have the conclusion. □

Note that in Lemma 3.6 we have the following identity $A^-B^-X_0ABX_0^- = X_1X_0^- = (A')^-(B')^-X_0A'B'X_0^-$.

**Lemma 3.7.** *Let $A, B, X, Y \in \mathcal{W}(E_n)$ be reduced words (possibly empty) such that $X$ and $Y$ are non-empty, $Y^-A^-B^-YX^-ABX \neq e$, $Y^-A^-B^-Y$ and $X^-ABX$ are reduced words, and the reduced word of $Y^-A^-B^-YX^-ABX$ is cyclically reduced.*

*If $Y^-A^-B^-YX^-ABX \in F_m$, then*

1. *$Y^-A^-B^-Y, X^-ABX \in F_m$, or*
2. *$Y^-A^-B^-YX^-ABX$ is equal to $c_s$ or $c_s^-$ for some $s$ such that $lh(s) = m$ and $s$ is binary branched.*

*Proof.* If $YX^-$ is reduced, then $Y^-A^-B^-YX^-ABX$ is cyclically reduced. By an argument analyzing the head and the tail of $Y^-$ and $X$ we can see $Y^-A^-B^-Y, X^-ABX \in F_m$.

Otherwise, in the cancellation of $Y^-A^-B^-YX^-ABX$ the leftmost $Y^-$ or the rightmost $X$ is deleted. Since $Y^-A^-B^-YX^-ABX \neq e$ and $lh(Y^-A^-B^-Y) = 2lh(Y)+lh(AB)$ and $lh(X^-ABX) = 2lh(X)+lh(AB)$, $lh(X) \neq lh(Y)$. We suppose that $lh(X) > lh(Y)$, i.e. the head of $Y^-$ is deleted. Then we have $X \equiv ZY$ for a non-empty word $Z$.

We first analyze the reduced word of $A^-B^-Z^-ABZ$, where $A^-B^-$ is deleted. The head part of $Z^-AB$ is $BA$. Applying Lemma 3.6 for $X_0 \equiv Z^-$ and $X_1$ repeatedly, we have reduced words $A_0$ and $B_0$ such that $Z^-A_0B_0Z$ is reduced, $Z^-A_0B_0 \equiv B_0A_0X_1$ for some $X_1$, $A_0^-B_0^-Z^-A_0B_0Z = A^-B^-Z^-ABZ$, $A, B \in \langle Z, A_0, B_0\rangle$ and $lh(B_0) < lh(Z)$.

It never occurs that both $A_0$ and $B_0$ are empty, but one of $A_0$ and $B_0$ may be empty. If $B_0 = \emptyset$, interchange the role of $A_0$ and $B_0$ and by Lemma 3.6 we can assume $B_0$ is non-empty and $lh(B_0) < lh(Z)$.

First we deal with the case $A_0$ is empty. Since the leftmost $B_0^-$ is deleted in the reduction of $B_0^-Z^-B_0Z$, we have non-empty $Z_0$ such that $Z \equiv Z_0B_0^-$ and have a reduced word $Z_0^-B_0Z_0B_0^-$ with $Z_0^-B_0Z_0B_0^- = B_0^-Z^-B_0Z$. Since the leftmost $Y^-$ is deleted in the reduction of $Y^-B_0^-Z^-B_0ZY$ and $Z_0^-B_0Z_0B_0^-Y$ is reduced, $Z_0^-B_0Z_0B_0^-$ is cyclically reduced and hence the reduced word of $Y^-A^-B^-YX^-ABX$ is a cyclical transformation of $Z_0^-B_0Z_0B_0^-$. By the fact that $Y$ is the head part of $B_0^-Z^-B_0ZY$, $Y$ is of the form $(Z_0^-B_0Z_0B_0^-)^kY_0$ where $Y_0Y_1 \equiv Z_0^-B_0Z_0B_0^-$ for some non-empty $Y_1$ and $k \geq 0$.

If $Y_0$ is empty, we have $Y^-A^-B^-YX^-ABX = Z_0^-B_0Z_0B_0^-$. If one of $Z_0$ and $B_0$ is not small, then $Z_0, B_0 \in F_m$ by Lemma 2.13 and we have $Y^-A^-B^-Y, X^-ABX \in F_m$ by Lemma 3.6 and the fact $Y = (Z_0^-B_0Z_0B_0^-)^k$. Otherwise, i.e., when of $Z_0$ and $B_0$ are small, $Y^-A^-B^-YX^-ABX = Z_0^-B_0Z_0B_0^-$ is equal to $c_s$ or $c_s^-$ for some $s$ such that $lh(s) = m$ and $s$ is binary branched by Lemma 2.14.

If $Y_0 \equiv Z_0^-$, $Y_0 \equiv Z_0^-B_0$ or $Y_0 \equiv Z_0^-B_0Z_0$, the argument is similar to the case that $Y_0$ is empty. Otherwise $Y_0$ ends somewhere in the middle of one of the words $Z_0^-$, $B_0$, $Z_0$ or $B_0^-$. Since the arguments are similar, we only deal with the case that $Y_0 \equiv Z_0^-B_1$ where $B_1B_2 \equiv B_0$ for non-empty $B_1$ and $B_2$. Then $Y^-A^-B^-YX^-ABX = B_2Z_0B_2^-B_1^-Z_0^-B_1$ and hence $B_2Z_0B_2^-, B_1^-Z_0^-B_1 \in F_m$ by Lemma 3.5 (4). Let $Z_1$ be a cyclically reduced word such that $Z_0 \equiv U^-Z_1U$. Then $Z_1, B_2U^-, UB_1 \in F_m$ by Lemma 2.11. Now

$$\begin{aligned}
Y^-Z_0Y &= B_1^-Z_0(B_1B_2Z_0^-B_2^-B_1^-Z_0)^kZ_0(Z_0^-B_1B_2Z_0B_2^-B_1^-)^kZ_0^-B_1\\
&= (B_1^-Z_0B_1B_2Z_0^-B_2^-)^kB_1^-Z_0B_1(B_2Z_0B_2^-B_1^-Z_0^-B_1)^k,\\
Y^-B_0Y &= B_1^-Z_0(B_1B_2Z_0^-B_2^-B_1^-Z_0)^kB_1B_2(Z_0^-B_1B_2Z_0B_2^-B_1^-)^kZ_0^-B_1\\
&= B_1^-Z_0B_1(B_2Z_0^-B_2^-B_1^-Z_0B_1)^kB_2Z_0^-B_1(B_2Z_0B_2^-B_1^-Z_0^-B_1)^k.
\end{aligned}$$

Hence $Y^- Z_0 Y, Y^- B_0 Y \in F_m$. Since $Z = Z_0 B_0^-$ and $A, B$ are elements of the subgroup $\langle Z, B_0 \rangle$ generated by $Z$ and $B_0$, we have $Y^- ABY, X^- A^- B^- X \in F_m$.

Next we suppose that $A_0$ is non-empty. We have $k > 0$ and $A_1$ and $A_2$ such that $Z^- \equiv B_0 A_1 A_2$, $A_0 \equiv (A_1 A_2)^k A_1$, $X_1 \equiv A_2 A_1 B_0$. Since $X^- AB \equiv UX_1$ for some $U$ and $X^- ABZ$ is reduced, $X_1 Z \equiv A_2 A_1 B_0 A_2^- A_1^- B_0^-$ is a reduced word. By the assumption a reduced word of $Y^- A_2 A_1 B_0 A_2^- A_1^- B_0^- Y$ is cyclically reduced and $A_2 A_1 B_0 A_2^- A_1^- B_0^- Y$ is reduced, hence $X_1 Z \equiv A_2 A_1 B_0 A_2^- A_1^- B_0^-$ is cyclically reduced and the reduced word of $Y^- A_2 A_1 B_0 A_2^- A_1^- B_0^- Y$ is given by a cyclical transformation of $A_2 A_1 B_0 A_2^- A_1^- B_0^-$. Hence $Y \equiv (A_2 A_1 B_0 A_2^- A_1^- B_0^-)^k Y_0$ where $k \geq 0$ and $A_2 A_1 B_0 A_2^- A_1^- B_0^- \equiv Y_0 Y_1$ for some $Y_1$.

For instance the reduced word of $Y^- A_2 A_1 B_0 A_2^- A_1^- B_0^- Y$ is of the form $B_0 A_2^- A_1^- B_0^- A_2 A_1$ or $B_2 A_2^- A_1^- B_2^- B_1^- A_2 A_1 B_1$ where $B_0 \equiv B_1 B_2$. By Lemma 3.5 (4) or (3) respectively we conclude $A_1, A_2, B_0 \in F_m$ or $A_1, A_2, B_1, B_2 \in F_m$ which implies $Y^- ABY, X^- A^- B^- X \in F_m$. □

## 4. Proof of Theorem 1.1

**Lemma 4.1.** *For every grope group $G^S$ the following hold:*

*If $e \neq [u, v] \in F_m$ and at least one of $u$ and $v$ does not belong to $F_m$, then $[u, v]$ is conjugate to $c_s$ or $c_s^-$ in $F_m$ for some $s$ such that $lh(s) = m$ and $s$ is binary branched.*

*Proof.* We have $n > m$ such that $u, v \in F_n$. It suffices to show the lemma in case that the reduced word for $[u, v]$ is cyclically reduced. For, suppose that we have the conclusion of the lemma in the indicated case. Let $[u, v] \in F_m$ and $[u, v] = XYX^-$ where $XYX^-$ is a reduced word and $Y$ is cyclically reduced. Then we have $[X^- uX, X^- vX] = X^- [u, v]X = Y$. On the other hand $X, Y \in F_m$ by Lemma 2.11. By the assumption at least one of $X^- uX$ and $X^- vX$ does not belong to $F_m$. Since $[u, v]$ is conjugate to $Y$ in $F_m$, we have the conclusion.

Let $u, v \in F_n$ such that $[u, v] \neq e$ and the reduced word for $[u, v]$ is cyclically reduced. There exist a cyclically reduced non-empty word $V_0 \in \mathcal{W}(E_n)$ and a reduced word $X \in \mathcal{W}(E_n)$ such that $v = X^- V_0 X$ and the word $X^- V_0 X$ is reduced. Let $U_0$ be a reduced word for $uX^-$. Since $V_0$ is a cyclically reduced word, at least one of $U_0 V_0$ and $V_0 U_0^-$ is reduced. When $U_0 V_0$ is reduced, there exist $k \geq 0$ and reduced words $Y, A, B$ such that $U_0 \equiv Y^- AV_0^k$, $V_0 \equiv BA$, and $Y^- ABY$ is reduced. When, however, $V_0 U_0^-$ is reduced, there exist $k \geq 0$ and reduced words $Y, A, B$ such that $U_0 \equiv Y^- A(V_0^-)^k$, $V_0 \equiv BA$, and $Y^- ABY$ is reduced. In both bases $uvu^{-1} = Y^- ABY$ and $v = X^- BAX$. Note that $AB$ and $BA$ are cyclically reduced.

We analyze the reduction procedure of $Y^- ABYX^- A^- B^- X$ in the three cases.

(Case 0): $X$ and $Y$ are empty.

In this case both $A$ and $B$ are non-empty and we can use Lemma 3.1. Using (1.1) and (1.2) alternately and (1.3) possibly as the last step we obtain the reduced word $A_0ZB_0A_0^-Z^-B_0^-$ of $ABA^-B^-$. Now there are two possibilities: If one of $A_0, Z, B_0$ is not small, by (1) and (2) of Lemma 3.5 $A_0$, $B_0$, $Z \in F_m$; by applying Lemma 3.1 repeatedly we get $A, B \in F_m$. If, however, $A_0$, $B_0$, $Z$ are not small, by Lemma 2.14 one of $A_0$, $B_0$, $Z$ is empty and $[u,v] = c_s$ or $[u,v] = c_s^-$ for some binary branched $s$ with $lh(s) = m$ .
(Case 1): Exactly of $X$ and $Y$ is empty.

Since the arguments are symmetric, we only deal with the case $Y$ is empty and $X$ is not empty. Therefore at least one of $A$ and $B$ is non-empty. As before we use Lemmas 3.2, 3.3, 3.4 to get the reduced word $A_0UVA_0^-CU^-V^-C^-$ of $ABX^-A^-B^-X$. Depending on the possibility that some of the above words are empty we apply one of (2), (3) or (4) of Lemma 3.5 to get $A$, $U$, $V$, $C \in F_m$. By applying Lemmas 3.2, 3.3, 3.4 repeatedly we get $A, B \in F_m$, which implies $u, v \in F_m$, or $[u,v] = c_s$ etc. as in (Case 0).
(Case 2): Both $X$ and $Y$ are non-empty.

This follows directly from Lemma 3.7. Only in this case we use the assumption that the reduced word of $Y^-ABYX^-A^-B^-X$ is cyclically reduced. □

**Lemma 4.2.** *Let $F$ be the free group generated by the set $B$ and $a, b \in B$ be distinct elements. If $[a,b] = [u,v]$ for $u, v \in F$, then neither $u$ nor $v$ belongs to the commutator subgroup of $F$.*

*Proof.* Since $a, b$ are generators, $[a,b] \notin [F,[F,F]]$ (Theorem 11.2.4,[7]) and the conclusion follows. □

**Lemma 4.3.** *Let $F$ be the free group generated by the set $B$ and $a, b \in B$ be distinct. If $c, d \in \{\beta, \beta^- : \beta \in B\}$ and $[a,b] = [x^{-1}cx, y^{-1}dy]$ for $x, y \in F$, then $c, d \in \{a, a^-, b, b^-\}$ and moreover $c \in \{a, a^-\}$ iff $d \in \{b, b^-\}$, and $c \in \{b, b^-\}$ iff $d \in \{a, a^-\}$.*

*Proof.* Using the canonical projection to the subgroup $\langle a, b\rangle$ we see that $c, d \in \{a, a^-, b, b^-\}$. To see the remaining part it suffices to show that if $c = a$, and $d = a$ or $a^-$, then $[a,b] \neq [x^{-1}cx, y^{-1}dy]$ for any $x, y$.

We show that $aba^-b^-$ is not cyclically equivalent to the reduced word of $[x^{-1}cx, y^{-1}dy]$. For this purpose we may assume $x = e$. We only deal with $d = a$. We have a reduced word $Y$ such that $y^{-1}ay = Y^-aY$ and $Y^-aY$ is reduced. (Note that $y \neq Y$ is possible.) The head of $Y$ is not $a$ nor $a^-$, since $Y^-aY$ is reduced. When the tail of $Y$ is $a$ or $a^-$, we choose $n \geq 0$ so that $Y \equiv Za^n$ or $Y \equiv Z(a^-)^n$ respectively and $n$ is maximal. Then $Z$ is non-empty. Now $aZ^-aZa^-Z^-a^-Z$ is a cyclically reduced word which is cyclically equivalent to $aY^-aYa^-Y^-a^-Y$. Since $aZ^-aZa^-Z^-a^-Z$ is not cyclically equivalent to $aba^-b^-$, we have the conclusion. □

*Proof of* Theorem 1.1. Let $h : G^{S_0} \to G^S$ be a nontrivial homomorphism. Let $c_s = c_s^{S_0}$, $d_t = c_t^S$, $F_n = F_n^S$, and $E_n = E_n^S$. Then there exists $s_* \in S_0$

such that $h(c_{s_*})$ is nontrivial (clearly for every finite sequence $s$ starting with $s_*$ also $h(c_s)$ is nontrivial).

We have $n$ such that $h(c_{s_*}) \in F_n$. Since $F_n$ is free, $\mathrm{Im}(h)$ is not included in $F_n$ and hence there exists $\varsigma \in S_0$ starting with $s_*$ and such that $h(c_\varsigma) \in F_n$, but $h(c_{\varsigma 0}) \notin F_n$ or $h(c_{\varsigma 1}) \notin F_n$. Then by Lemma 4.1 we have $d_\tau \in E_n$ such that $h(c_\varsigma)$ is conjugate to $d_\tau$ or $d_\tau^-$ and $\tau$ is binary branched.

Moreover, Lemma 2.10 implies that neither $h(c_{\varsigma 0})$ nor $h(c_{\varsigma 1})$ belongs to $F_n$. We show the following by induction on $k \in \mathbb{N}$:

(1) For $u \in Seq(\underline{2})$ with $lh(u) = k$

   (a) $h(c_{\varsigma u})$ is conjugate to $d_{\tau v}$ or $d_{\tau v}^-$ in $F_{n+k}$ and $\tau v$ is binary branched for some $v \in Seq(\underline{2})$ with $lh(v) = k$;
   (b) Neither $h(c_{\varsigma u 0})$ nor $h(c_{\varsigma u 1})$ belongs to $F_{n+k}$;

(2) For every $v \in Seq(\underline{2})$ with $lh(v) = k$ there exists $u \in Seq(\underline{2})$ such that $lh(u) = k$ and $h(c_{\varsigma u})$ is conjugate to $d_{\tau v}$ or $d_{\tau v}^-$ in $F_{n+k}$.

We have shown that this holds when $k = 0$.

Suppose that (1) and (2) hold for $k$. Let $lh(u) = k$ and $h(c_{\varsigma u}) = [h(c_{\varsigma u 0}), h(c_{\varsigma u 1})]$ is conjugate to $d_{\tau v} = [d_{\tau v 0}, d_{\tau v 1}]$ or $d_{\tau v}^- = [d_{\tau v 1}, d_{\tau v 0}]$ in $F_{n+k}$.

We claim $h(c_{\varsigma u 0}) \in F_{n+k+1}$. To show this by contradiction, suppose that $h(c_{\varsigma u 0}) \notin F_{n+k+1}$. Apply Lemma 4.1 to $F_{n+k+1}$, then we have $[h(c_{\varsigma u 0}), h(c_{\varsigma u 1})]$ is conjugate to $d_t$ or $d_t^-$ with $lh(t) = n + k + 1$ in $F_{n+k+1}$, which is impossible since $[h(c_{\varsigma u 0}), h(c_{\varsigma u 1})] \in [F_{n+k+1}, F_{n+k+1}]$. Similarly we have $h(c_{\varsigma u 1}) \in F_{n+k+1}$.

On the other hand, neither $h(c_{\varsigma u 0})$ nor $h(c_{\varsigma u 1})$ belongs to $[F_{n+k+1}, F_{n+k+1}]$ by Lemma 4.2. Hence at least one of $h(c_{\varsigma u 00})$ and $h(c_{\varsigma u 01})$ does not belong to $F_{n+k+1}$ and consequently neither $h(c_{\varsigma u 00})$ nor $h(c_{\varsigma u 01})$ belongs to $F_{n+k+1}$ by Lemma 2.10.

Hence $h(c_{\varsigma u 0})$ is conjugate to $d_t$ or $d_t^-$ with $lh(t) = n+k+1$ by Lemma 4.1. Similarly, $h(c_{\varsigma u 1})$ is conjugate to $d_{t'}$ or $d_{t'}^-$ with $lh(t') = n + k + 1$. Since $h(c_{\varsigma u}) = [h(c_{\varsigma u 0}), h(c_{\varsigma u 1})]$ is conjugate to $d_{\tau v} = [d_{\tau v 0}, d_{\tau v 1}]$ or $d_{\tau v}^- = [d_{\tau v 0}, d_{\tau v 1}]$ in $F_{n+k+1}$, $h(c_{\varsigma u 0})$ and $h(c_{\varsigma u 1})$ are conjugate to $d_{\tau v j}$ or $d_{\tau v j}^-$ for some $j \in \underline{2}$ and for each $j \in \underline{2}$ the element $d_{\tau v j}$ is conjugate to exactly one of $h(c_{\varsigma u 0})$, $h(c_{\varsigma u 1})$, $h(c_{\varsigma u 0})^-$ or $h(c_{\varsigma u 1})^-$ by Lemma 4.3. Hence (1) and (2) hold for $k + 1$. Now we have shown the induction step and finished the proof. □

**Remark 4.4.** *Though the conclusion of Theorem 1.1 is rather simple, embeddings from $G^{S_0}$ into $G^S$ may be complicated. In particular the group of automorphisms of $G^{S_0}$ is uncountable. We explain this more precisely. Observe that*

$$[dc^-d^-, dcd^-c^-d^-] = dc^-d^-dcd^-c^-d^-dcd^-dcdc^-d^- = cdc^-d^- = [c, d].$$

*For $T \subseteq S_0$ we define $\varphi_T(c_{s0}) = c_{s1}c_{s0}^-c_{s1}^-$ and $\varphi_T(c_{s1}) = c_{s1}c_{s1}^-c_{s0}^-c_{s1}^-$ for $s \in T$ and $\varphi_T(c_{s0}) = c_{s0}$ and $\varphi_T(c_{s1}) = c_{s1}$ for $s \notin T$. Then we have automorphisms $\varphi_S \neq \varphi_T$ for distinct subsets $S$ and $T$ of $S_0$. Hence we have uncountable many automorphism on $G^{S_0}$. In particular this shows that* *there are uncountably many self-homotopy equivalences of the minimal grope since for every one of the countably many attached 2-cells we have a choice of more than one possibility to extend the self-homotopy equivalence.*

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