Difference and differential equations (from a point of view of asymptotic cones)

Katsuya Eda

Department of Mathematics, Waseda University, Tokyo 169-8555, JAPAN

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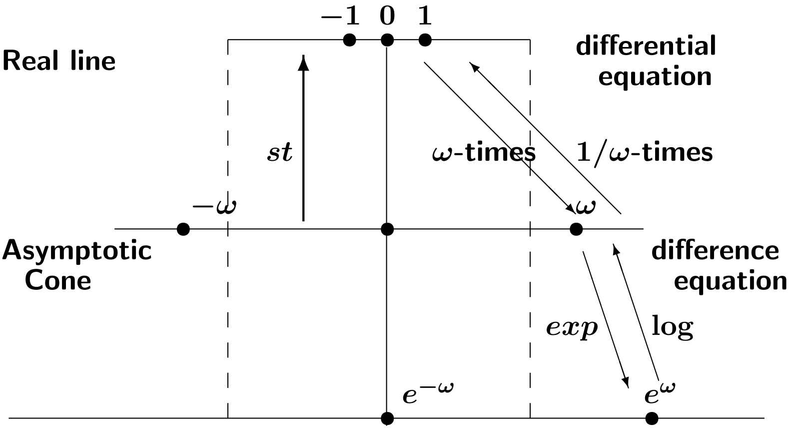
Some notions in asymptotic phenomenons

1. Asymptotic cones (M. Gromov)
   reformulated by van den Dries and Wilkie
2. Toropical geometry (O. Viro)
3. Ultra-discrete analysis (D. Takahashi)
4. A. V. Maslov correspondence

Scopes of views



Ultra-discrete — Toropical geometry — Maslov Correspondence

$\mathbb{Z}$ for $\mathbb{R}$

Let ω be a fixed hyperfinite natural number and let $\varepsilon = 1/\omega$.

$$\begin{aligned} {}^{\#}\mathbb{R} &= \{x \in {}^{*}\mathbb{R} \mid \text{ there exists } n \in \mathbb{N} \text{ such that } |x| < n\omega\}, \\ {}^{\#}\mathbb{Z} &= {}^{\#}\mathbb{R} \cap {}^{*}\mathbb{Z}. \end{aligned}$$

For a metric space (X, d) choose $x_0 \in X$ or $x_0 \in {}^{*}X$

$$ {}^{\#}X = \{x \in {}^{*}X \mid \text{ for some } n \in \mathbb{N},\ d(x_0, x) \leq n\omega\}\}. $$

For $x, y \in {}^{*}X$

$$ x \sim_{\#} y \leftrightarrow \text{ for all } n \in \mathbb{N},\ d(x, y) \leq \omega/n. $$

For $x, y \in {}^{\#}\mathbb{R}$, we use $|\ |$ as metric and use $\sim_{\#}$.

Then, for $x, y \in {}^{*}X$, $\varepsilon d(x, y) \sim 0$ and $x \sim_{\#} y$ are equivalent and $d(x, y) \in {}^{\#}\mathbb{R}$.

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Asymptotic cone

This set of equivalence classes ${}^{\#}X/\sim_{\#}$ is due to M. Gromov and called an asymptotic cone, and expressed as $(C_\infty X, d_\#)$. Our construction is seemingly different from a well-known one, but they are actually the same.

Theorem (M. Gromov): For $x, y \in {}^{\#}X$ let

$$d_\#([x],[y]) = \mathrm{st}(\varepsilon d(x,y)).$$

Then ${}^{\#}X/\sim_{\#}$ is a complete metric space under $d_\#$.

proof. It is easy to see that $d_\#$ is a metric.
To show the completeness, let $([x_n] : n \in \mathbb{N})$ be a Cauchy sequence in $C_\infty X$, i.e. particularly let

$$d_\#([x_m],[x_n]) < 1/m \quad (m \leq n).$$

We want to show the existence of a convergent point assuming

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Discete and continuous

Next we investigate ${}^{\#}\mathbb{R}$ under the reflection of $(\mathbb{R}, +, \cdot, \leq)$. It is easy to see that ${}^{\#}\mathbb{R}$ and ${}^{\#}\mathbb{Z}$ are subgroups of ${}^{*}\mathbb{R}$ under the operation $+$.
But, since $\omega^2 \notin {}^{\#}\mathbb{R}$, ${}^{\#}\mathbb{R}$ is not closed under the multiplication.
Let

$$x \cdot_\varepsilon y = \varepsilon xy.$$

Then $({}^{\#}\mathbb{R}, +, \cdot_\varepsilon)$ is an associative ring. For n-variable real function f, let

$$f_\omega(u_1, \cdots, u_n) = \omega f(\varepsilon u_1, \cdots, \varepsilon u_n).$$

In the following we define in the form for one variable functions, but the n-variable case is similar.

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In the following we define in the form for one variable functions, but the n-variable case is similar.

If f is a totally defined or bounded real continuous function, $\overline{\mathbb{R}}$ is closed under f and hence $^{\#}\mathbb{R}$ is closed under f_ω.

Theorem:
If f is a totally defined or bounded real continuous function, then for $r \in \mathbb{R}$ we have

$$\begin{aligned} f(r) &= \mathrm{st}(\varepsilon f_\omega(\lfloor \omega r \rfloor)) \\ &= \mathrm{st}(\varepsilon \lfloor f_\omega(\lfloor \omega r \rfloor) \rfloor). \end{aligned}$$

This theorem implies that a totally defined or bounded real continuous function corresponds to a function from $^{\#}\mathbb{Z}$ to $^{\#}\mathbb{Z}$ which is in the non-standard universe, i.e. so called an internal function.

Differences and differentials

Therem: If a real function f is differentiable at $r \in \mathbb{R}$,

$$\begin{aligned} f'(r) &= \mathrm{st}(f_\omega(\omega r + 1) - f_\omega(\omega r)) \\ &= \mathrm{st}(f_\omega(\lfloor \omega r \rfloor + 1) - f_\omega(\lfloor \omega r \rfloor)). \end{aligned}$$

Proof. Since

$$f_\omega(\omega r + 1) = \omega f(\varepsilon(\omega r + 1)) = \omega f(r + \varepsilon),$$

by the standard argument of non-standard analysis we have

$$\begin{aligned} f'(r) &= \mathrm{st}\left(\frac{f(r+\varepsilon) - f(r)}{\varepsilon}\right) \\ &= \mathrm{st}(\omega(f(r+\varepsilon) - f(r))) \\ &= \mathrm{st}(f_\omega(\omega r + 1) - f_\omega(\omega r)). \end{aligned}$$

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Proof continued

Next let $0 \le a < 1$ be such that $\lfloor \omega r \rfloor = \omega r - a$. In case $a = 0$ the conclusion is obvious and we assume $a > 0$. Since

$$\begin{aligned} f_\omega(\lfloor \omega r \rfloor + 1) &= \omega f(\varepsilon(\omega r - a + 1)) = \omega f(r + (1-a)\varepsilon) \\ f_\omega(\lfloor \omega r \rfloor) &= \omega f(r + (-a)\varepsilon), \end{aligned}$$

we have

$$\begin{aligned} & f_\omega(\lfloor \omega r \rfloor + 1) - f_\omega(\lfloor \omega r \rfloor) \\ = \ & \omega(f(r + (1-a)\varepsilon) - f(r) - (f(r + (-a)\varepsilon) - f(r)). \end{aligned}$$

Then, the standard part of the left term is equal to

$$\begin{aligned} &= \operatorname{st}(\ (1-a)\ \frac{f(r + (1-a)\varepsilon) - f(r)}{(1-a)\varepsilon} \\ &\qquad + a\ \frac{f(r + (-a)\varepsilon) - f(r)}{(-a)\varepsilon}\ ) \\ &= \operatorname{st}(1-a) f'(r) + \operatorname{st}(a) f'(r) \\ &= f'(r). \end{aligned}$$

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According this theorem we see the relationship between differences and differentials is not only in analogy, but is in a logical relationship with formal descriptions.

A part of the following is a colaboration with Shigeaki Nagamachi of Tokushima University.

As well-known, solutions of constant coefficient linear differece equations and constant coefficient linear differential equations are given by roots of their characteristic equations.

A constant coefficient linear differece equation

$$\sum_{i=0}^{n} a_i f(x+i) = 0$$

and a constant coefficient linear differential equation

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Let ρ be a root of this equation. Then ρ^x is a solution of the difference equation and $e^{\rho x}$ is a solution of the differential equation.

Here is a question why Napier's constant e appears and how the solutions are related.

We'll answer these using our setting.

First, we express i-th differential function $f^{(i)}(x)$ by using $f_\omega(x)$.

For $x \in \mathbb{R}$

$$f^{(i)}(x) \sim \omega^{i-1} \sum_{j=0}^{i} {}_iC_j(-1)^{i-j} f_\omega(\omega x + j)$$

holds.

Proof can be done by rather straight-forward induction.

Since

$$f^{(0)}(x) = f(x) = \varepsilon f_\omega(\omega x) = \omega^{-1} {}_0C_0 f_\omega(\omega x),$$

we have the case $i = 0$.

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Rewriting

Since by induction hypothesis

$$\begin{aligned}
f^{(i)}(x+\varepsilon) &\sim \omega^{i-1}\sum_{j=0}^{i} {}_iC_j(-1)^{i-j}f_\omega(\omega(x+\varepsilon)+j) \\
&\sim \omega^{i-1}\sum_{j=1}^{i+1} {}_iC_{j-1}(-1)^{i+1-j}f_\omega(\omega x+j) \\
&\sim \omega^{i-1}(f_\omega(\omega x+i+1) \\
&\qquad +\sum_{j=1}^{i} {}_iC_{j-1}(-1)^{i+1-j}f_\omega(\omega x+j)),
\end{aligned}$$

Rewriting continued

we have

$$
\begin{aligned}
f^{(i+1)}(x) &\sim (f^{(i)}(x+\varepsilon) - f^{(i)}(x))/\varepsilon \\
&\sim \omega^i (f_\omega(\omega x + i + 1) \\
&\qquad\qquad + \sum_{j=1}^{i} {}_iC_{j-1}(-1)^{i+1-j} f_\omega(\omega x + j) \\
&\quad + \sum_{j=1}^{i} {}_iC_j(-1)^{i+1-j} f_\omega(\omega x + j) + (-1)^{i+1} f_\omega(\omega x) \\
&\sim \omega^i \sum_{j=0}^{i+1} {}_{i+1}C_j(-1)^{i+1-j} f_\omega(\omega x + j).
\end{aligned}
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Consider a constant coefficient linear differential equation

$$\sum_{i=0}^{n} a_i f^{(i)}(x) = 0,$$

(where $a_n = 1$). Now we have

$$\sum_{i=0}^{n} a_i \omega^{i-1} \sum_{j=0}^{i} {}_iC_j (-1)^{i-j} f_\omega(\omega x + j) \sim 0.$$

We try to find g such that

$$f(x) = \mathrm{st}(\varepsilon g(\lfloor \omega x \rfloor))$$

for $x \in \mathbb{R}$ and

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Suppose the factorization of the characteristic equation of a given differential equation is

$$\prod_{i=0}^{n}(x-\rho_i)=\sum_{i=0}^{n}a_i x^i=0.$$

According to the preceding transformation of a differential equation to a difference equation we have

$$\sum_{i=0}^{n}a_i\omega^{i-1}\sum_{j=0}^{i}{}_iC_j(-1)^{i-j}g(x+j)=0.$$

The characteristic equation of this difference equation is

$$\begin{aligned} 0 &= \sum_{i=0}^{n}a_i\omega^{i-1}\sum_{j=0}^{i}{}_iC_j(-1)^{i-j}X^j \\ &= \sum_{i=0}^{n}a_i\omega^{i-1}(X-1)^i \end{aligned}$$

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Suppose the factorization of the characteristic equation of a given differential equation is

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According to the preceding transformation of a differential equation to a difference equation we have

$$\sum_{i=0}^{n}a_i\omega^{i-1}\sum_{j=0}^{i}{}_iC_j(-1)^{i-j}g(x+j)=0.$$

The characteristic equation of this difference equation is

$$\begin{aligned}0 &= \sum_{i=0}^{n}a_i\omega^{i-1}\sum_{j=0}^{i}{}_iC_j(-1)^{i-j}X^j\\ &= \sum_{i=0}^{n}a_i\omega^{i-1}(X-1)^i\end{aligned}$$

the last term $\sum_{i=0}^{n} a_i \omega^{i-1}(X-1)^i$ is equal to

$$\varepsilon \sum_{i=0}^{n} a_i \omega^i (X-1)^i$$

and we have

$$\begin{aligned} 0 = \sum_{i=0}^{n} a_i(\omega(X-1))^i &= \prod_{i=0}^{n}(\omega(X-1)-\rho_i) \\ &= \omega^{n+1}\prod_{i=0}^{n}(X-(1+\varepsilon\rho_i)) \end{aligned}$$

Finally we have

$$\prod_{i=0}^{n}(X-(1+\varepsilon\rho_i)) = 0.$$

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That is, a root of this characteristic equation is $1 + \varepsilon\rho_i$. For instance, when $1 + \varepsilon\rho$ is an m-multiple root, as well-known, the solution is given as

$$\sum_{k=0}^{m-1} C_k x^k (1 + \varepsilon\rho)^x.$$

Therefore, if we put

$$g(x) = \sum_{k=0}^{m-1} C_k x^k (1 + \varepsilon\rho)^x,$$

we have

$$\sum_{i=0}^{n} a_i \omega^{i-1} \sum_{j=0}^{i} {}_iC_j (-1)^{i-j} g(\lfloor \omega x \rfloor + j) = 0.$$

Using

$$(1 + \varepsilon\rho)^{\omega x} \sim (e^{\rho})^x = e^{\rho x},$$

for $C_k = \varepsilon^{k-1} c_k$ $(c_k \in \mathbb{R})$ we check

$$f(x) = \mathrm{st}(\varepsilon g(\lfloor \omega x \rfloor)).$$

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Finish

The last term will be really a well-known solution of the original constant coefficient linear differential equation. It goes

$$|(\omega x)^m - \lfloor \omega x \rfloor^m| = |\omega x - \lfloor \omega x \rfloor| (\Sigma_{k=0}^{m-1} (\omega x)^{m-1-k} \lfloor \omega x \rfloor^k)$$

and

$$\varepsilon \lfloor \omega x \rfloor \sim x, \; 0 \leq |\omega x - \lfloor \omega x \rfloor| < 1.$$

By induction we have

$$\varepsilon^m (\omega x)^m \sim \varepsilon^m \lfloor \omega x \rfloor^m.$$

Then

$$(1 + \varepsilon\rho)^{\lfloor \omega x \rfloor} \sim (1 + \varepsilon\rho)^{\omega x} \sim (e^\rho)^x = e^{\rho x}$$

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and hence

$$
\begin{aligned}
\mathrm{st}(\varepsilon g(\lfloor \omega x \rfloor)) &= \sum_{k=0}^{m-1} c_k \ \mathrm{st}(\varepsilon^k \lfloor \omega x \rfloor^k (1+\varepsilon\rho)^{\lfloor \omega x \rfloor}) \\
&= \sum_{k=0}^{m-1} c_k x^k e^{\rho x}.
\end{aligned}
$$

Questions

What discrete subgroups of a Lie group make the Lie group as their asymptotic ones?

Y. de Cornulier and P. de la Harpe, Metric geometry of locally compact groups, 2014.
Asymptotic cones of Lie groups and cone equivalences, Y. de Cornulier, 2009.

What discrete structures of discrete subgroups reflects the differential structures of the Lie groups?

Can we say more on relationship between the structures on discrete subgroups and those on the Lie groups?

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