Specker phenomenon in the uncountable case

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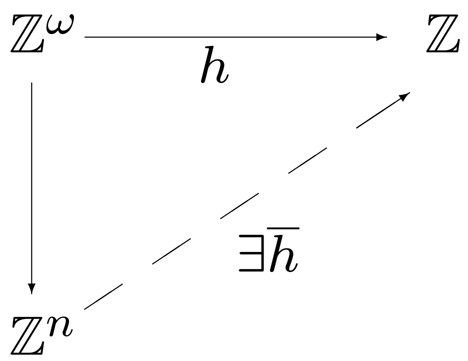
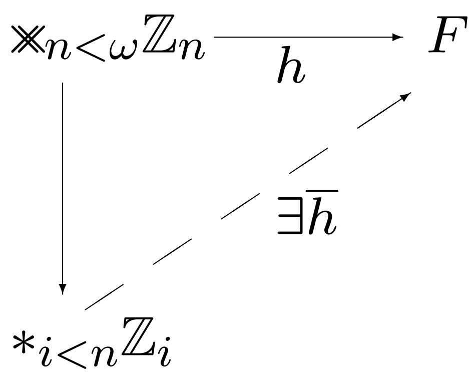
E. Specker (1950)
Given homomorphism
$h : \mathbb{Z}^\omega \to \mathbb{Z}$



G. Higman (1952)
Given homomorphism
$h : \circledast_{n<\omega}\mathbb{Z}_n \to F$

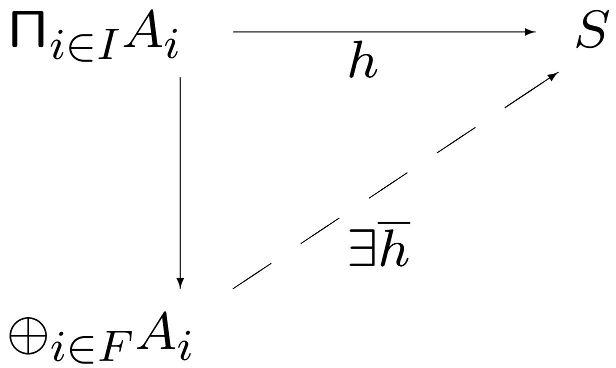


$\circledast_{n<\omega}\mathbb{Z}_n$ may be
$\lim_{\leftarrow}(*_{i<n}\mathbb{Z}_i, p_{mn} : m \le n < \omega)$.

Uncountable, commutative case

J. Łoś,   E. C. Zeeman (1955)

Let S be a slender abelian group and $h : \Pi_{i \in I} A_i \to S$.

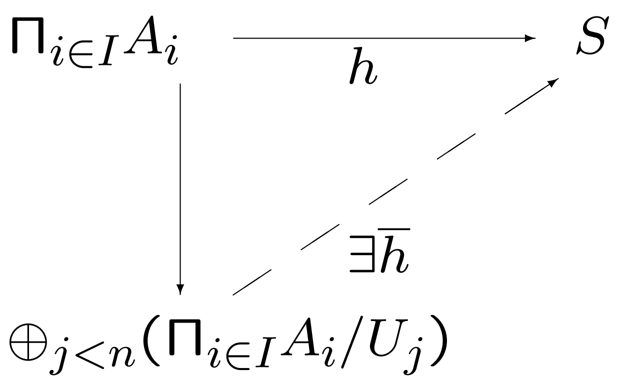


F is a finite subset of I, where the cardinality of I is less than the least measurable cardinal.

Uncountable, commutative case (continued)

K. E. (1982)

Let S be a slender abelian group and
$h : \prod_{i \in I} A_i \to S$ be a homomorphism.
Then there exist countably complete ultrafilters U_j:



Why does the least measurable cardinal appear?

Let $h : C(X, A) \to B$, where h is a homomorphism A and B are abelian groups. A subset S is a support of h, if $f|S = 0$ implies $h(f) = 0$. In many cases supports form a filter. Can we have the minimal support? If X is compact, it is easy to have it. In the case of abelian groups we have the minimal support in the following case. X is $\mathbb{N}$-comact, i.e. X is a closed subspace of a direct product of the discrete space $\mathbb{N}$ and B is a slender group.

A discrete space X is $\mathbb{N}$-comact, if and only if the cardinality of X is less than the least measurable cardinals.

Unrestricted free product $\lim_{\leftarrow}(*_{i\in F}G_i, p_{EF} : E \Subset F \subset I)$

Complete free product (= free complete product) $⋇_{i\in I}G_i$ consisting elements expressed by words.

Assume $G_i \cap G_j = \{e\}$ for $i \neq j$.
A word is $W : \overline{W} \to \cup_{i\in I}G_i$, where $\overline{W}$ is linearly ordered and $\{\alpha \in \overline{W} : W(\alpha) \in G_i\}$ is finite for each $i \in I$.

(Difference)
Let $x_n = \delta_0\delta_1\delta_0 \cdots \delta_0^{-i}\delta_i\delta_0^i \cdots \delta_0^{-n}\delta_n\delta_0^n$ for $1 \leq n < \omega$. Then $p_{nn+1}(x_n) = x_{n-1}$ and $(x_n : n < \omega) \in \lim_{\leftarrow}(*_{i=0}^m\mathbb{Z}_i : m < \omega)$, but $(x_n : n < \omega) \notin ⋇_{n<\omega} *_{i=0}^n \mathbb{Z}_i$.

IMPORTANT ASPECT: There exists a unique reduced word of each word.

Difficulty involving non-commutativity:

For disjoint index sets I and J,

$\Pi_{i\in I\cup J}A_i \simeq \Pi_{i\in I}A_i \oplus \Pi_{i\in J}A_i$,   but

$\lim_{\leftarrow}(*_{i\in F}G_i, p_{EF} : E \Subset F \subset I \cup J)$

$\not\simeq \lim_{\leftarrow}(*_{i\in F}G_i, p_{EF} : E \Subset F \subset I) * \lim_{\leftarrow}(*_{i\in F}G_i, p_{EF} : E \Subset F \subset J)$,

nor

$$⋇_{i\in I\cup J}G_i \quad \not\simeq \quad ⋇_{i\in I}G_i * ⋇_{i\in J}G_i.$$

Uncountable, non-commutative case

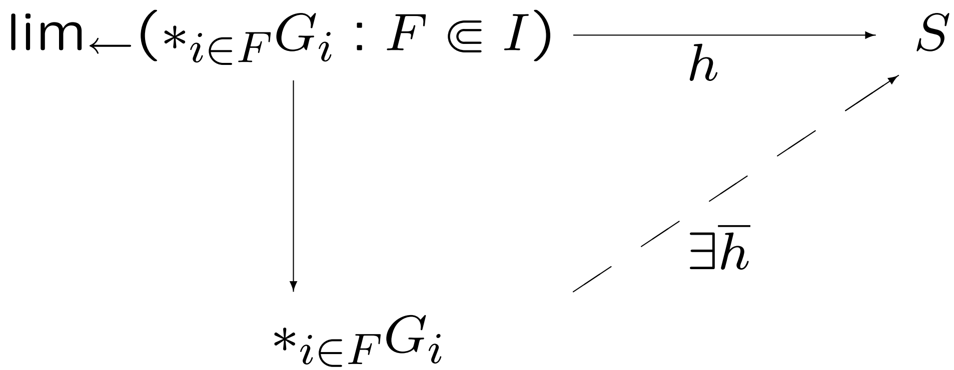
S. Shelah - K. E. (2002)

Let S be an n-slender group and $h : \text{lim}_{\leftarrow}(*_{i \in F} G_i, p_{EF} : E \subset F \Subset I) \to S$ be a homomorphism. For $X \subseteq I$, let $p_X : \text{lim}_{\leftarrow}(*_{i \in F} G_i, p_{EF} : E \Subset F \subset I) \to \text{lim}_{\leftarrow}(*_{i \in F} G_i, p_{EF} : E \subset F \Subset X)$ be the canonical projection.

Then there exist countably complete ultrafilters U_j such that $h = h \circ p_{X_1 \cup \cdots \cup X_n}$ for every $X_j \in U_j$.

(Continued)

Consequently if the cardinality of I is less than the least measurable cardinal, there exists a finite subset F of I such that



There are questions about non-commutative ultraproducts
$$*_{j<n}\lim_{\leftarrow}(*_{i\in F}G_i : F \Subset I)/U.$$
What are the structures when G_i s are copies of $\mathbb{Z}$.

Failure in Uncountable, non-commutative case

S. Shelah - L. Struengmann (2001)

If the index set I is uncountable, there exists a homomorphism $h : \circledast_{i \in I} \mathbb{Z}_i \to \mathbb{Z}$ such that $h(*_{i \in I} \mathbb{Z}_i) = \{e\}$ but h is non-trivial. A tail T of a word W is a non-emty word such that $W \equiv W_0 T$ for some word W_0. Two words V, W are tail-equivalent, if V and W have the same tail. Let σ be the tail-equivalence class containing a word W of uncountable cofinality. We can define a homomorphism h_σ to $\mathbb{Z}$ according to the appearances of tails of a reduced word W and heads of W^-.

A group S is n-slender, if for an arbitrary homomorphism $h : \circledast_{n<\omega}\mathbb{Z}_n \to S$ the Higman diagram holds, i.e. the non-commutative Specker phenomenon holds.

Description of homomorphisms
$h : \circledast_{i\in I} G_i \to S$.

For a word $W \in \mathcal{W}(G_i : i \in I)$, define supp(W) $= \{$i $\in$ I : W(α) $\in$ G$_i$ for some $\alpha\}$. A family of words $\mathcal{F}$ is *overlapping*, if any infinite subfamily of $\mathcal{F}$ has an infinite subfamily $\mathcal{G}$ such that $\bigcap\{$supp(W) : W $\in \mathcal{G}\}$ is non-empty. A family of uncountable cofinal types of reduced words is overlapping, if each family of words consisting of all the representatives is overlapping.

(Claim1): Let W be a reduced word of uncountable cofinality. Then there exists a tail T of W such that for each tail T' of T $h(T) = h(T')$.

Otherwise, we have subwords V_n such that $T \equiv V_0 V_1 \cdots V_n \cdots T_0$ and $h(V_n)$ s are non-trivial.
We have a homomorphism $\varphi : \circledast_{n<\omega} \mathbb{Z}_n \to \circledast_{i \in I} G_i$ such that $\varphi(\delta_n) = V_n$ and $h \circ \varphi(\delta_n) \neq e$, which contradicts the n-slenderness of S.

For h let $\mathcal{T}_h$ be the set of all tail-equivalence classes τ of reduced words of uncoutable cofinalities such that $h(T)$ is non-trivial for a sufficiently small tail $T \in \tau$.

(Claim1): There exist only finitely many i s such that $h \,|\, G_i$ is non-trivial.

(Claim2): $\mathcal{T}_h$ is an overlapping family.

For a tail-equivalence class σ σ^- denotes the head equivalent class. For an overlapping family $\mathcal{T}$ of tail-equivalence classes and $f : \mathcal{T} \to S \setminus \{e\}$ we define a homomorphism $h_f : \circledast_{i \in I} G_i \to S$ as follows:

Let W be a reduced word. Find gaps τ or τ^- for $\tau \in \mathcal{T}$ in W. Remark that there exist only finitely many ones. Hence according to $f(\tau)$ and $f(\tau)^{-1}$ we define $h_f(W)$.

Theorem: A homomorphism $h : \circledast_{i \in I} G_i \to S$ is given by a natural amalgamation of h_f and finitely many $h_i : G_i \to S$ ($i \in F \Subset I$), where h_f is given by an overlapping family $\mathcal{T}$ and a map from $\mathcal{T}$ to $S \setminus \{e\}$. Moreover $\mathcal{T}$ defines a free subgroup H of $\circledast_{i \in I} G_i$ where $*_{i \in F} G_i * U$ is a retract and h is factored through this subgroup.

To show the latter statement, we choose $W_\tau \in \tau \in \mathcal{T}$ so that W_τ does not contain elements in $\bigcup_{i \in F} G_i$ and any tail of W_τ is not a subword of W_σ for $\sigma \neq \tau$. Then $\{W_\tau : \tau \in \mathcal{T}\}$ freely generates a subgroup of $\circledast_{i \in I} G_i$, i.e. $*_{\tau \in \mathcal{T}} \langle W_\tau \rangle$ is a free subgroup. The retraction $r : \circledast_{i \in I} G_i \to *_{i \in F} G_i * *_{\tau \in \mathcal{T}} \langle W_\tau \rangle$ is defined as follows: For a reduced word W find elements in $\bigcup_{i \in F} G_i$ and also gaps τ or τ^- for $\tau \in \mathcal{T}$ in W, then leave elements in $\bigcup_{i \in F} G_i$ and correspond W_τ or W_τ^- according to the ordering of gaps in W.

Question of non-commutative ultraproducts

Investigate

$$\varprojlim(*_{i\in F}\mathbb{Z}_i : F \Subset I)/U,$$

when U is a non-principal countably complete ultrafilter on I.

When U induces measures on $[I]^n$ s, the group is of the cardinality $2^{\aleph_0}$. What is the cardinality in general?