Lecture1: Algebraic topology of one-dimensional continua
Čech homotopy groups of one-dimensional continua

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Well-known facts

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Theorem 1 (Well-known). Let X be a one-dimensional continuum. Then, the Čech homology group $\check{H}_1(X)$ is isomorphic to a free abelian group of finite rank or the direct product of countable copies of the integer group $\mathbb{Z}^\omega \cong \check{H}_1(\mathbb{H})$.

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In addition if X is locally connected, i.e. X is a Peano continuum, the Čech homotopy group (shape group) $\check{\pi}_1(X)$ is isomorphic to a free group of finite rank or the canonical inverse limit of free groups of finite rank which is isomorphic to $\check{\pi}_1(\mathbb{H})$.

continued

Why are the assumptions different (local connectivity)?
Answer: Under the local connectivity, we have arbitrary finer finite coverings consisting of connected open sets. Hence the bonding homomorphisms between free groups of finite rank become surjective.
Since every subgroup of a free abelian group of finite rank is also a free abelian group of finite rank, in the abelian case we can reform the inverse sysytem so that every bonding homomorphism is surjective. But, a subgroup of a free group of finite rank may not be a free group of finite rank (The commutator subgroup of the free group of rank 2 is a free group of countable rank).

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The Čech homotopy group of the Hawaiian earring (the canonical inverse limit of free groups of finite rank).

(2) $\varprojlim(G_n, p_n : n < \omega)$ where $G_n = *_{i<n}\mathbb{Z}_i$ and $p_n : G_{n+1} \to G_n$ is the projection such that $p_n \mid *_{i<n}\mathbb{Z}_i = \mathrm{id}$ and $p_n(\mathbb{Z}_n) = \{e\}$;

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Mimick the construction of (2) and use copies of F_ω instead of those of $\mathbb{Z}$.

(5) $\varprojlim(G_n, p_n : n < \omega)$ where $G_0 = F_\omega$ and $G_{n+1} = G_n * F_{\omega n}$ where $F_{\omega n}$ is a copy of F_ω, $p_n : G_{n+1} \to G_n$ is the projection such that $p_n \mid G_n = \mathrm{id}$ and $p_n(F_{\omega n}) = \{e\}$.

Caution about inverse limits

It scarecely happens that

$$\varprojlim_{n\to\infty} G_n * H_n \cong \varprojlim_{n\to\infty} G_n * \varprojlim_{n\to\infty} H_n.$$

In particular

$$\varprojlim_{n\to\infty} (*_{0\le i<n}\mathbb{Z}_i, p_n) \not\cong \mathbb{Z} * \varprojlim_{n\to\infty} (*_{1\le i<n}\mathbb{Z}_i, p_n).$$

This is far from the case of the fundamental group of the Hawaiian earring, where we have free groups of finite rank as its free factors.

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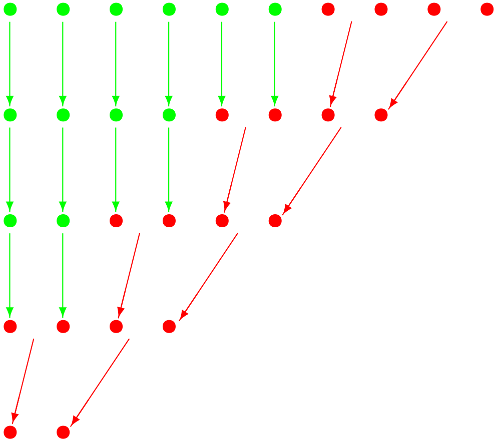
Conclusion

As SHAPE theorists know it, the SHAPE groups are NOT reliable.

How is the free group of countable rank realized?

Consider the following injective inverse sequence. The limit is a subgroup of the free group of rank 2!



Basic facts about free groups

Theorem 3 (Well-known, but nontrivial fact).
Let $h : F_0 \to F_1$ be a surjective homomorphism between free groups.
Then, $Ker(h)$ is a free factor of F_0, i.e.

$$F_0 = Ker(h) * H \text{ for some } H.$$

Theorem 3 implies that the inverse limit of free groups of finite rank is isomorphic to one of groups (1)-(5).

Non-isomorphicness of groups in the list

Groups in (1) and (3) are countable.
A free groups which are homomorphic images of the group (2) is of finite rank by the Higman theorem (Specker phenomenon).

$\oplus_\omega \mathbb{Z} \oplus \mathbb{Z}^\omega$ is the homomorphic image of the group (4) (G_4).
$(\oplus_\omega \mathbb{Z})^\omega$ is the homomorphic image of the group (5) (G_5).

These groups are in the Reid-class [EkM] and non-isomorphic.
Let $R_{\mathbb{Z}}(A) = \bigcap\{Ker(h) | h \in \mathrm{Hom}(A, \mathbb{Z})$. Then

$$Ab(G_4)/R_{\mathbb{Z}}(G_4) \cong \oplus_\omega \mathbb{Z} \oplus \mathbb{Z}^\omega, \quad Ab(G_5)/R_{\mathbb{Z}}(G_5) \cong (\oplus_\omega \mathbb{Z})^\omega.$$

Torsionfree algebraically compact abelian groups

Well-known facts:

(1)(due to Kaplansky): It is a direct sum of the divisible subgroup ($\cong \oplus_I \mathbb{Q}$) and the direct product of A_p for primes p, where A_p is the p-adic completion of a free abelian group.

(2) The algebraical compactness is equivalent to the pure-injectivity.

Less-known fact (due to Dugas-Goebel): A is algebraically compact if and only if $U(A) = UU(A)$ and $A/U(A)$ is complete under $\mathbb{Z}$-adic topology, where $U(A) = \bigcap_{n\in\mathbb{N}} n!\, A$.

$$(n+1)! | a_{n+1} - a_n \; (n \in \mathbb{N}) \rightarrow$$

$$\exists a_\infty (\;\; (n+1)! \, | \, a_\infty - a_n \; (n \in \mathbb{N}) \;)$$

If A is torsionfree, $U(A) = UU(A)$ holds.

Secret Fact

It easy to apply these to Wild Topology. If sizes of loops or maps converge to zero, we can add infinitely many meaningful ones. For given a_n with $(n+1)! \mid a_{n+1} - a_n$, find loops b_n with

$$(n+1)!b_n = a_{n+1} - a_n$$

such that b_n is of small sizes or is equal to the sum of loops small sizes as homology classes.
Intuitively put

$$a_\infty = \Sigma_{n=1}^{\infty}(n+1)!b_n + a_1.$$

One necessary trick here is to make the divisibility in the noncommutative stage.

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