Lecture2: Fundamental groups and singular homology groups of one-dimensional continua

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Fundamental groups and singular homology groups

The fundamental groups of one-dimensional Peano continua determine the homotopy types of them [E2]. Particularly, the fundamental groups of everywhere non-semi-locally simply connected one-dimensional Peano continua determine the homeomorphism types of them [E1]. Therefore, the fundamental groups of one-dimensional Peano continua are abundant. The singular homology groups H_1 are the abelianizations of the fundamental groups and consequently they possibly may be less abundant. The following shows that they are not only less abundant, but scare, and they have the same simple classification as the Čech homology groups and shape groups.

Singular homology groups

Let X be a one-dimensional Peano continuum. Then the singular homology group $H_1(X)$ is isomorphic to a free abelian group of finite rank or the singular homology group of the Hawaiian earring $H_1(\mathbb{H})$

$$\cong \mathbb{Z}^{\omega} \oplus \oplus_{\mathbf{c}}\mathbb{Q} \oplus \Pi_{p:\mathrm{prime}} A_p,$$

where ω is the least infinite ordinal, $\mathbf{c}$ is the cardinality of the continuum and A_p is the p-adic completion of the free abelian group of rank $\mathbf{c}$ [EK].

Torsionfree algebraically compact abelian groups

Well-known facts:

(1)(due to Kaplansky): It is a direct sum of the divisible subgroup ($\cong \bigoplus_I \mathbb{Q}$) and the direct product of A_p for primes p, where A_p is the p-adic completion of a free abelian group.

(2) The algebraical compactness is equivalent to the pure-injectivity.

Less-known fact (due to Dugas-Goebel): A is algebraically compact if and only if $U(A) = UU(A)$ and $A/U(A)$ is complete under $\mathbb{Z}$-adic topology, where $U(A) = \bigcap_{n\in\mathbb{N}} n!\, A$.

$$(n+1)! | a_{n+1} - a_n \ (n \in \mathbb{N}) \longrightarrow$$

$$\exists a_\infty(\ \ (n+1)! \,|\, a_\infty - a_n \ (n \in \mathbb{N}) \ )$$

If A is torsionfree, $U(A) = UU(A)$ holds.

Secret Fact

It easy to apply these to Wild Topology. If sizes of loops or maps converge to zero, we can add infinitely many meaningful ones. For given a_n with $(n+1)! \mid a_{n+1} - a_n$, find loops b_n with

$$(n+1)!b_n = a_{n+1} - a_n$$

such that b_n is of small sizes or is equal to the sum of loops small sizes as homology classes.

Intuitively put

$$a_\infty = \Sigma_{n=1}^{\infty}(n+1)!b_n + a_1.$$

One necessary trick here is to make the divisibility in the noncommutative stage.

References

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