Lecture3: Infinite-sheeted connected covering spaces over solenoids

Katsuya Eda

Department of Mathematics, Waseda University, Tokyo 169-8555, JAPAN

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Solenoids

Solenoids Σ are one dimensional connected compact abelian groups distinct from the circle group $\mathbb{R}/\mathbb{Z}$.
This is obtained as the inverse limit of $\mathbb{R}/\mathbb{Z}$ with finite-sheeted covering maps and also obtained as the Pontrjagin dual of subgroups of the rational group$\mathbb{Q}$.
Finite sheeted covering maps over Σ having connected total spaces are equivalent to homomorphisms, which are the Pontrjagin dual of inclusion maps between subgroups of $\mathbb{Q}$.
Through the Pontrjagin duality the total spaces of finite sheeted covering homomorphisms corresponds to the finite index super groups in $\mathbb{Q}$.

Covering maps and covering homomorphisms

Natural question:
Is it possible to define a topological group structure on a total space X so that a covering map $f : X \to Y$ over a topological group Y becomes a homomorphism of topological groups?

1(Well-known). Yes, if Y is a pathwise connected, locally pathwise connected group and X is a pathwise connected space.
2(Less-known). Yes, if Y is a compact connected group, X is connected and f is finite-sheeted.

Covering maps and overlays

R. Fox introduced overlays in 1972.

An overlay is a covering map with an additional property.

Let Y be a connected space, $f : X \to Y$ a continuous map and S a set. For open coverings $\mathcal{B}$ of Y and $\mathcal{A} = \{A_B^\sigma : B \in \mathcal{B}, \sigma \in S\}$ of X, $(\mathcal{A}, \mathcal{B})$ is an S-sheeted covering pair for $f : X \to Y$, if

(C1) $f^{-1}(B) = \bigcup_{\sigma \in S} A_B^\sigma$, $B \in \mathcal{B}$

(C2) $A_B^\sigma \cap A_B^\tau = \emptyset$, for $\sigma, \tau \in S \sigma \neq \tau; B \in \mathcal{B}$;

(C3) $f|_{A_B^\sigma} : A_B^\sigma \to B$ is a homeomorphism for each $A_B^\sigma \in \mathcal{A}$.

$(\mathcal{A}, \mathcal{B})$ is an S-sheeted overlay pair for f, if $(\mathcal{A}, \mathcal{B})$ is an S-sheeted covering pair for f, $\mathcal{B}$ is a normal covering and

(C4) If $B, B' \in \mathcal{B}$ and $B \cap B' \neq \emptyset$, then every $\sigma \in S$ admits a unique $\sigma' \in S$ such that $A_B^\sigma \cap A_{B'}^{\sigma'} \neq \emptyset$.

Overlays and covering homomorphisms

Theorem 1. Let Y be a compact connected group, X a connected space and let $f : X \to Y$ be a covering map. X admits a topological group structure such that f is a covering homomorphism if and only if f is an overlay map.

Theorem 2. Let X be a connected space and $f : X \to \Sigma$ be an infinite-sheeted covering map over a solenoid Σ. Then X does not admit a topological group structure such that f is a covering homomorphism.

Presentation of a solenoid

Let P be a sequence of primes which is related to a solenoid Σ_P.
$P = \langle p_0, p_1, \cdots \rangle \sim Q = \langle q_0, q_1, \cdots \rangle$ equivalent, written $P \sim Q$, if we have the same sequence by deletions of finitely many primes from each sequence.
(Well-known) Σ_P and Σ_Q are homeomorphic(isomorphic) if and only if $P \sim Q$.
Define a P- adic group $\mathbb{J}_P$ and
a quotient space $\mathbb{J}_P \times [0, 2\pi]/\sim$ which is homeomorphic to Σ_P. A member of $\mathbb{J}_P$ is formally written as $\Sigma_{n=0}^{\infty} u_n \Pi_{i=0}^{n-1} p_i$ and $\mathbb{J}_P$ is a compact, totally disconnected topological abelian group. When P is the constant p sequence, $\mathbb{J}_P$ is the p-adic integer group $\mathbb{J}_p$ for a prime p.

Basic strategy of constructing covering

References

[EM] K. Eda and V. Matijević, Covering maps over solenoids which are not covering homomorphisms, Fund. Math., in press.