HOMOTOPY TYPES OF ONE-DIMENSIONAL PEANO CONTINUA

KATSUYA EDA

Abstract. Let X and Y be one-dimensional Peano continua. If the fundamental groups of X and Y are isomorphic, then X and Y are homotopy equivalent. Every homomorphism from the fundamental group of X to that of Y is a composition of a homomorphism induced from a continuous map and a base point change isomorphism.

1. Introduction and definitions

In this paper we prove:

Theorem 1.1. *Let X and Y be one-dimensional Peano continua. If the fundamental groups of X and Y are isomorphic, then X and Y are homotopy equivalent.*

Theorem 1.2. *Let X be a one-dimensional Peano continuum, Y a one-dimensional metric space and $x \in X$ and $y \in Y$. For each homomorphism $h : \pi_1(X, x) \to \pi_1(Y, y)$ there exists a continuous map $f : X \to Y$ and a path q from $f(x)$ to y such that $h = \varphi_q \circ f_*$, where φ_q is the base point change isomorphism.*

Corollary 1.3. *Let X and Y be one-dimensional Peano continua and $f : X \to Y$ a continuous map. If f induces an isomorphism between the fundamental groups of X and Y, then f is a homotopy equivalence between X and Y.*

It seems that the first theorem was conjectured in the middle of 1990's and the author heard of it from G. Conner. If spaces X and Y are locally simply connected in addition to the conditions in Theorems 1.1 and 1.2, then X and Y are homotopy equivalent to finite graphs and their fundamental groups are free groups and consequently the conclusions of Theorems 1.1 and 1.2 are obvious. But in general it is very non-trivial. On the other hand, if X and Y are not locally

2000 *Mathematics Subject Classification.* Primary:55Q20, 55Q70; Secondary:54E45.

Key words and phrases. fundamental group, one-dimensional, Peano continuum.

simply connected at any point, the isomorphism h between the fundamental groups of X and Y induces a homeomorphism $\tilde{h}$ between X and Y [7], which is not so trivial. Hearing Theorem 1.1 U. Karimov asked the author whether Corollary 1.3 holds and this is an affirmative answer to his question. Since our proofs are on the extended line of proofs in [7] some of whose notions are uncommon ones, we restate definitions.

For $a \leq b$, a continuous map $f : [a, b] \to X$ is called a path from $f(a)$ to $f(b)$. The points $f(a)$ and $f(b)$ are called the initial point and the terminal point of f respectively. When $a = b$, the path f is said to be *degenerate*. A loop f is a path with $f(a) = f(b)$. For a path $f : [a, b] \to X$, f^- denotes a path such that $f^-(s) = f(a + b - s)$ for $a \leq s \leq b$. Two paths $f : [a, b] \to X, g : [c, d] \to X$ are *equivalent*, denoted by $f \cong g$, if there exists a homeomorphism $\varphi : [a, b] \to [c, d]$ such that $\varphi(a) = c, \varphi(b) = d$ and $f = g \circ \varphi$. Two paths $f : [a, b] \to X$ and $g : [c, d] \to X$ are *homotopic*, denoted by $f \sim g$, if there exists a continuous map H whose domain is the quadrangle in the plane with the vertices $(a, 0), (b, 0), (c, 1)$ and $(d, 1)$ such that

$$\begin{cases} H(s,0) = f(s) & \text{for } a \leq s \leq b, \\ H(s,1) = g(s) & \text{for } c \leq s \leq d, \\ H((1-t)a + tc, t) = f(a) = g(c) & \text{for } 0 \leq t \leq 1, \\ H((1-t)b + td, t) = f(b) = g(d) & \text{for } 0 \leq t \leq 1. \end{cases}$$

The homotopy class containing a path f is denoted by $[f]$. The homotopy defined above is usually called "a homotopy relative to end points." We drop the term "relative to end points" for simplicity.

A path $f : [a, b] \to X$ is *reduced* if each subloop of f is not null-homotopic, that is, for each pair $u < v$ with $f(u) = f(v)$, $f \restriction [u, v]$ is not null-homotopic. Note that a constant map is reduced if and only if it is degenerate. For paths $f : [a, b] \to X$ and $g : [c, d] \to X$ with $f(b) = g(c)$, fg denotes the concatenation of f and g, that is, a path from $[a, b + d - c]$ to X such that $fg(s) = f(s)$ for $a \leq s \leq b$ and $fg(s) = g(s - b + c)$ for $b \leq s \leq b + d - c$. A loop f is *cyclically reduced*, if ff is reduced. An *arc* A between points x and y is a subspace of X which is homeomorphic to the unit interval $[0, 1]$ whose end points are x and y. The Hawaiian earring is the plane continuum $\mathbb{H} = \bigcup_{n=1}^{\infty} \{(x, y) : (x - 1/n)^2 + y^2 = 1/n^2\}$ and o denotes the origin $(0, 0)$. Each simple closed curve of the Hawaiian earring is parametrized as follows: $\mathbf{e}_n(t) = ((1 + \cos(\pi + 2\pi t))/n, \sin(\pi + 2\pi t)/n)$ for $1 \leq n < \omega, 0 \leq t \leq 1$.

Let O^X be the subset consisting of elements x such that X is locally simply connected at x. Then O^X is an open subset of X for a locally path-connected space X. We define $X^w = X \setminus O^X$. For a homomorphism $h : \pi_1(X, x) \to G$, let X_h^w be the set of all points $x_0 \in X$ such that, for each neighborhood U of x_0, there exists a loop f in U such that $h(\varphi_g([f])) \neq e$ for some path g from x_0 to x. Let $O_h^X = X \setminus X_h^w$. We remark that in this definition the choice of a path g does not effect anything. When h is injective, we have $X_h^w = X^w$. We also remark that X_h^w is closed and O_h^X is open for a locally path-connected space X.

2. Reduction of one-dimensional Peano continua

A subset A of a space X is called an *open arc*, if A is open in X and is homeomorphic to the open interval $(0, 1)$. An open arc in a Peano continuum has at least one and at most two end points and so we denote these by A^0 and A^1. We remark that $A^0 = A^1$ may happen.

The next theorem has been proved in the master thesis of M. Meilstrup [15]. Since our proof will be modified to prove Theorem 1.2, we prove this precisely.

Theorem 2.1. (*M. Meilstrup* [15]) *Every one-dimensional Peano continuum is homotopy equivalent to a one-dimensional Peano continuum X such that X is a finite connected graph or O^X is an at most countable union of open arcs the end points of which belong to X^w.*

Our proof is a modification of the proof of [7, Theorem 1.2], particularly that of the implication (3)⇒(2), and hence we recommend the reader to review [7, Section 4] and to proceed.

A metric space (X, ρ) is *uniformly locally connected* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\rho(x, y) < \delta$ then x and y are contained in a connected open set of diameter less than ε. We refer the reader to [14, Section 2.1.1] for the brick partion and facts around it. An important fact is: if O is a uniformly locally connected, connected open set in a Peano continuum, then $\overline{O}$ is also a Peano continuum.

A *partition* $\mathcal{P}$ of a space is a pair-wise disjoint family of finitely many connected open sets such that $\bigcup \mathcal{P}$ is dense. A partition $\mathcal{P}$ is of order 2 if $\overline{P_1} \cap \overline{P_2} \cap \overline{P_3} = \emptyset$ for distinct $P_1, P_2, P_3 \in \mathcal{P}$. A partition $\mathcal{P}$ is a *brick* partition if $\mathcal{P}$ consists of regular open sets and $int(\overline{P} \cup \overline{Q})$ is uniformly locally connected for each $P, Q \in \mathcal{P}$. Consequently each element of $\mathcal{P}$ is uniformly locally connected.

For a subset S of X, the diameter of S is denoted by $\mathrm{diam}(S)$, i.e. $\mathrm{diam}(S) = \sup\{\rho(x, y) : x, y \in S\}$ and $\mathrm{Mesh}(\mathcal{P}) = \max\{\mathrm{diam}(P) : P \in \mathcal{P}\}$. Since our construction is a reformation of the proof of [14, Theorem 2.9], we state their result in a suitable form to our case.

Proposition 2.2. [14, Theorem 2.9] *Let X be a one-dimensional Peano continuum, K a 0-dimensional closed set and $x_0, x_1, \cdots, x_n \in X$. Then, for every $\varepsilon > 0$, there exists a brick partition $\mathcal{P}$ of order 2 such that*

(1) $K \subseteq \bigcup \mathcal{P}$;
(2) *if $x_i \neq x_j$, then there are distinct $P_i, P_j \in \mathcal{P}$ such that $x_i \in P_i, x_j \in P_j$;*
(3) $\mathrm{Mesh}(\mathcal{P})$ *is less than* ε;
(4) *the boundary of each member of $\mathcal{P}$ is 0-dimensional.*

Proof of Theorem 2.1. Let X be a one-dimensional Peano continuum. If $X^w = \emptyset$, then the argument below shows that X is homotopy equivalent to a finite connected graph. We therefore assume that $X^w \neq \emptyset$.

Our first goal is to construct dendrites D_n and open arcs A_i so that

(1) D_n is a dendrite such that $D_n \cap X^w = \{x_n\}$, $D_n \setminus \{x_n\} \subseteq O^X$ and $\partial D_n \subseteq \bigcup_i \partial A_i \cup \{x_n\}$;
(2) $A_i \subseteq O^X$;
(3) $D_m \cap D_n \subseteq X^w$ for $m \neq n$ and $A_i \cap A_j = \emptyset$ for $i \neq j$;
(4) $X^w \cup \bigcup_n D_n \cup \bigcup_i A_i$ is a strong deformation retract of X;
(5) $\lim_{n\to\infty} \mathrm{diam}(D_n) = 0$ and $\lim_{i\to\infty} \mathrm{diam}(A_i) = 0$.

For it we construct brick partitions $\mathcal{P}_m$, open arcs A_i, parts of dendrites D_{mn}, and points y_{mn} by induction.

In the 0-step we let $\mathcal{P}_0 = \{ \text{X} \}$, but we do not define A_i and so on. After the m-th step, we have finitely many points y_{mn} which enumerate the boundary of $\bigcup\{\overline{P} : P \in \mathcal{P}_m, P \cap X^w \neq \emptyset\}$. First we work in each $\overline{P}$ for $P \in \mathcal{P}_m$. Applying Proposition 2.2 to a 0-dimensional closed set ∂P and points y_{mn} in ∂P, we have a brick partition $\mathcal{P}_P$ of $\overline{P}$ satisfying

(1) $\mathcal{P}_P$ is of order 2 and $\mathrm{Mesh}(\mathcal{P}_P) < 1/(m+1)$;
(2) if $\overline{Q} \cap X^w = \emptyset$ for $Q \in \mathcal{P}_P$, $\overline{Q}$ is simply connected;
(3) if $\overline{Q} \cap X^w = \emptyset$ for $Q \in \mathcal{P}_P$, $\overline{Q} \cap \overline{Q'}$ is at most one point for $Q' \in \mathcal{P}_P$ with $Q' \neq Q$.
(4) If $y_{mn} \in \overline{P}$, then there exists $Q \in \mathcal{P}_P$ such that $y_{mn} \in Q$ and $\overline{Q} \cap X^w = \emptyset$.

Next let $\mathcal{P}_{m+1}$ be the family

$$\{Q \setminus \partial P \mid Q \in \mathcal{P}_P, P \in \mathcal{P}_m \text{ satisfying } \overline{P} \cap X^w \neq \emptyset\}.$$

Since ∂P does not separate any nonempty connected open set in $\overline{P}$, $\mathcal{P}_{m+1}$ is a partition of $\bigcup\{\overline{P} : P \in \mathcal{P}_m, \overline{P} \cap X^w \neq \emptyset\}$ and also a brick partition of it. Since $\partial P \subseteq \bigcup \mathcal{P}_P$, $\mathcal{P}_{m+1}$ is of order 2. Hence $\mathcal{P}_{m+1}$ is a brick partition of $\bigcup\{\overline{P} : P \in \mathcal{P}_m, \overline{P} \cap X^w \neq \emptyset\}$ which satisfies the following:

(1) $\mathcal{P}_{m+1}$ is of order 2 and $\mathrm{Mesh}(\mathcal{P}_{m+1}) < 1/(m+1)$;
(2) $\mathcal{P}_{m+1}$ refines the restriction of $\mathcal{P}_m$ to $\bigcup\{\overline{P} : P \in \mathcal{P}_m, \overline{P} \cap X^w \neq \emptyset\}$;
(3) if $\overline{Q} \cap X^w = \emptyset$ for $Q \in \mathcal{P}_{m+1}$, $\overline{Q}$ is simply connected;
(4) if $\overline{Q} \cap X^w = \emptyset$ for $Q \in \mathcal{P}_{m+1}$, $\overline{Q} \cap \overline{Q'}$ is at most one point for $Q' \in \mathcal{P}_{m+1}$ with $Q' \neq Q$.
(5) If $y_{mn} \in \overline{P}$ for $P \in \mathcal{P}_m$ with $\overline{P} \cap X^w \neq \emptyset$, then there exists $Q \in \mathcal{P}_{m+1}$ such that $Q \subseteq P$, $y_{mn} \in \overline{Q}$ and $\overline{Q} \cap X^w = \emptyset$.

For each $Q \in \mathcal{P}_{m+1}$ with $\overline{Q} \cap X^w = \emptyset$, ∂Q is finite and so we connect these points by arcs and have a finite tree T_Q which is a strong deformation retract of $\overline{Q}$ in $\overline{Q}$, since $\overline{Q}$ is a uniquely arcwise connected Peano continuum. Then $\bigcup\{T_Q : Q \in \mathcal{P}_{m+1}, \overline{Q} \cap X^w = \emptyset\} = G_{m+1}$ is a finite graph, where we consider branching points in T_Q and points in ∂Q as vertices. Then $G_{m+1} \cap \bigcup\{\overline{Q} : Q \in \mathcal{P}_{m+1}, \overline{Q} \cap X^w \neq \emptyset\}$ is a finite set, which we enumerate without repetition as $y_{m+1\,n}$.

For y_{ml} we have a unique $P \in \mathcal{P}_m$ such that $y_{ml} \in \overline{P}$ and $\overline{P} \cap X^w \neq \emptyset$. According to (5) in the construction of $\mathcal{P}_{m+1}$, every point y_{ml} is connected to at least one point $y_{m+1\,n}$ by an arc A so that $y_{m+1\,n} \in \overline{Q}$ for $Q \in \mathcal{P}_{m+1}$ with $Q \subseteq P$ and $\overline{Q} \cap X^w \neq \emptyset$, and $A \setminus \{y_{ml}\} \subseteq P$. By working in each P we have a tree $T_n^P \subseteq G_{m+1}$ so that y_{ml} are connected to $y_{m+1\,n}$ by T_n^P. According to this, if $y_{m+1\,n}$ is connected to some y_{ml}, then $y_{m+1\,n}$ belongs to a unique $P \in \mathcal{P}_m$. Therefore, there exists at most one P such that a tree T_n^P is connected to each $y_{m+1\,n}$ and call it T_n. When no T_n^P is connected to $y_{m+1\,n}$ we let T_n be the singleton of $y_{m+1\,n}$. In such a case $y_{m+1\,n}$ may belong to $\overline{P} \cap \overline{P'}$ for distinct $P, P' \in \mathcal{P}_m$. But there exists a unique $P \in \mathcal{P}_m$ such that $y_{m+1\,n} \in \overline{Q}$ for $Q \in \mathcal{P}_{m+1}$ with $Q \subseteq P$ and $\overline{Q} \cap X^w \neq \emptyset$. We work in the unique $P \in \mathcal{P}_m$ for such T_n in the following procedure. We remark that T_n s are pairwise disjoint.

Vertices in G_{m+1} which are in $\overline{P}$ may not belong to any T_n. Next we expand T_n s in G_{m+1} so that every vertex of G_{m+1} belongs to one of extensions of trees T_n. We want to control the sizes of expanded trees and so we work in each $\overline{P}$. By induction on n we construct a maximal tree T_n^* in $G_{m+1} \cap \overline{P}$ or $G_{m+1} \cap (\overline{P_0} \cup \overline{P_1})$ such that $T_n \subseteq T_n^*$, but $T_n^* \cap \bigcup_{k<n} T_k^* = \emptyset$ and $T_n^* \cap \bigcup_{k>n} T_k = \emptyset$. Inductively, we assume that if D_{ml} has been defined for y_{ml} then D_{ml} is connected to a unique T_n^* and we connect all such D_{ml} to T_n^*, thus forming $D_{m+1\,n}$. We remark that any D_{ml} may not be connected to some T_n^* and that T_n^* may be even a singleton of $y_{m+1\,n}$. Then, we add all open edges of $G_{m+1} \setminus \bigcup_n T_n^*$ to the list of A_i. We remark the size of A_i added in the $(m+1)$-st step is less than $1/(m+1)$.

After continuing every step we have D_{mk} which are contained in a unique $D_{m+1k'}$. Since our procedure of picking y_{mk} is done in each separated small area, a sequence consists of y_{mk} s converges one point in X^w. We enumerate these points as x_n and let the increasing union of D_{mk} for which y_{mk} converge to x_n and together with the singleton $\{x_n\}$ to be D_n, which is a dendrite by its construction. For each m, there are only finitely many D_n which contain y_{mk} for some k and, if D_n does not contain y_{mk} for any k, D_n is contained in some $\overline{P_0} \cup \overline{P_1}$ for $P_0, P_1 \in \mathcal{P}_m$ and hencediam$(D_n) \leq 2/m$. For each $P \in \mathcal{P}_m$ with $\overline{P} \cap X^w = \emptyset$, we have a strong deformation retraction from P to T_P. Since $\lim_{n\to\infty} \mathrm{Mesh}(\mathcal{P}_n) = 0$, we can take a union of strong deformation retractions of P to T_P as a strong deformation retraction of X to $X^w \cup \bigcup_n D_n \cup \bigcup_i A_i$. Now we have achieved the first goal.

Let $X_0 = X^w \cup \bigcup_n D_n \cup \bigcup_i A_i$. Since $x_m = x_n$ may happen for $m \neq n$, we form a union of D_n s when $x_m = x_n$. Then it still is a dendrite. Hence we suppose that $D_m \cap D_n = \emptyset$ for $m \neq n$. Let Z be the quotient space of X_0 obtained by regarding each D_n as one point. Since $\lim_{n\to\infty} \mathrm{diam}(D_n) = 0$, Z is a compact metrizable space (see [4, Propositions I.2.2 and I.2.3]). Since Z is a countable sum of one-dimensional closed sets X^w and $\overline{A_i}$ s, Z is one-dimensional by [13, 7.2.1 Theorem] and hence Z is a one-dimensional Peano continuum. The remaining task is to show that X_0 is homotopy equivalent to Z.

Let $f : X_0 \to Z$ be the quotient map. To define $g : Z \to X_0$, we take strong contractions $r_n : D_n \times [0,1] \to D_n$ such that $r_n(x_n, t) = x_n, r_n(u, 1) = x_n$ for each $u \in D_n$ and we take continuous maps $a_i : [0,1] \to \overline{A_i}$ so that $a_i(0) = A_i^0$ and $a_i(1) = A_i^1$ and $a_1 \restriction (0,1)$ is a homeomorphism.

We define $g(u) = u$ for $u \in X^w$ and so it suffices to define $g(u)$ for u in each A_i. There exists a unique n_0 such that $A_i^0 \in D_{n_0}$ and also a unique n_1 such that $A_i^1 \in D_{n_1}$. Define g on A_i by:

$$g(u) = \begin{cases} r_{n_0}(A_i^0, 1-3s) & \text{if } u = a_i(s) \text{ for } 0 \leq s \leq 1/3 \\ a_i(3s-1) & \text{if } u = a_i(s) \text{ for } 1/3 < s < 2/3 \\ r_{n_1}(A_i^1, 3s-2) & \text{if } u = a_i(s) \text{ for } 2/3 \leq s \leq 1 \,. \end{cases}$$

The continuity of g on $\bigcup_i A_i$ is obvious and so we consider of that on $x \in X^w$. For an open neighborhood U of $g(x) = x$ in X_0, choose a neighborhood U_0 of x in X_0 so that $\overline{U_0} \subseteq U$. Let $I_0 = \{i : A_i^0 \in D_n \text{ or } A_i^1 \in D_n, x_n \in U_0\}$. There exist at most finitely many $x_n \in U_0$ for which the set $D_n \cup \bigcup\{A_i : A_i^0 \in D_n \text{ or } A_i^1 \in D_n\}$ is not contained in U. For such an $x_n \in U_0$, consider the connected component C_n of $U \cap (D_n \cup \bigcup\{A_i : A_i^0 \in D_n \text{ or } A_i^1 \in D_n\})$ containing x_n. Then, there exist at most finitely many A_i such that $A_i^0 \in D_n$ or $A_i^1 \in D_n$ and $\overline{A_i} \not\subseteq C_n$. Collecting these we have an at most finite subset I_1 of I_0 such that $g(A_i) \subseteq U$ for all $i \in I_0 \setminus I_1$.

Since $(U_0 \cap X^w) \cup \bigcup_{i \in I_0} A_i$ is an open neighborhood of x in Z, just shrinking on A_i for $i \in I_1$ we have the desired neighborhood V_0 of x such that $g(V_0) \subseteq U$, i.e. g is continuous. Since it is comparatively easy to prove $f \circ g$ is homotopic to id_Z, we only prove that $g \circ f$ is homotopic to id_{X_0}.

Define $H : X_0 \times [0,1] \to X_0$ by:

$$\begin{aligned} H(x,0) &= x \text{ for } x \in X_0, \\ H(x,t) &= x \text{ for } x \in X^w, \\ H(x,t) &= r_n(x,t) \text{ for } x \in D_n, \end{aligned}$$

and

$$H(a_i(s),t) = \begin{cases} r_{n_{i0}}(A_i^0, t-s-2st), & \text{if } 0 \le s \le 1/3 \text{ and } s+2st-t < 0 \\ a_i(s+2st-t), & \text{if } 0 \le s \le 1/3 \text{ and } s+2st-t \ge 0 \\ a_i(s+2st-t), & \text{if } 1/3 \le s \le 2/3 \\ a_i(s+2st-t), & \text{if } 2/3 \le s \le 1 \text{ and } s+2st-t \le 1 \\ r_{n_{i1}}(A_i^1, s+2st-t-1), & \text{if } 2/3 \le s \le 1 \text{ and } s+2st-t > 1. \end{cases}$$

Then, $H(x,1) = g \circ f(x)$ for $x \in X_0$. The continuity of H on (x,t) for $x \in X^w$ is shown by a similar consideration as for the continuity of g. □

Remark 2.3. We remark a difference between the proof of [7, Theorem 1.2] and the above one. In the former case D_{mk} s and A_i s converge to one point automatically and the care for the sizes of connecting paths between y_{ml} and $y_{m+1\,n}$ is not necessary. But in the above case, if we do not take care, we might not be able to find the end point of D_n in X^w. So, we connect them in each $\overline{P}$ for $P \in \mathcal{P}_m$ separately.

3. Lemmas for paths

First we recall some contents from [7].

Lemma 3.1. [7, Lemma 5.1] *Let X be a first countable space and Y be one-dimensional metric space and $h : \pi_1(X,x) \to \pi_1(Y,y)$ be a homomorphism. Then, for $x_0 \in X_h^w$ there exists a unique point $y_0 \in Y$ which satisfies the following condition:*

> *for a path $p : [0,1] \to X$ with $p(0) = x_0$ and $p(1) = x$, there exists a unique path $q : [0,1] \to Y$ from $y_0 = q(0)$ to $y = q(1)$ up to homotopy which satisfies the following:*

for each continuous map $f : (\mathbb{H}, o) \to (X, x_0)$ there exists a continuous map $g : (\mathbb{H}, o) \to (Y, y_0)$ such that $h \circ \varphi_p \circ f_* = \varphi_q \circ g_*$.

Using this lemma we defined $\tilde{h} : X^w_h \to Y$ in [7, p.497]. We want to extend $\tilde{h}$ to X. For this purpose we recall the setting from [7, Section 6]. Here we generalize it a little.

For a one-dimensional space X, a point $x \in X$ and a subset S of X, $P(X)$ is the set of all paths in X, $P_x(X)$ is the set of all paths which terminate at x, $P_S(X)$ is the set of all paths which connect points in S, and $P_{S,x}(X)$ is the set of all paths which start from points in S and terminate at x, $P^h(X)$ is the set of all homotopy classes of paths in X, $RP(X)$ is the set of all reduced paths in X, $RP_x(X)$ is the set of all reduced paths in X which terminate at x, and $P^h_x(X)$ is the homotopy classes that are represented by the paths in $P_x(X)$. Since any path is homotopic to a reduced path (see [3]), there is a one-to-one correspondence between $P^h(X)$ (or $P^h_x(X)$) and the quotient of $RP(X)$ (or $RP_x(X)$) modulo the equivalence. According to our definition of homotopies between paths, homotopies are relative to end points, the initial point and the terminal point of the class $[p]$ are well-defined for a homotopy class $[p] \in P^h(X)$. If pq is a path for two paths p and q, $[p][q]$ is defined as $[pq]$. An element of $P^h(X)$ is *degenerate*, if it is the equivalence class of a degenerate path.

For an open set U containing the initial point of $[p]$, let $O(U, [p]) = \{[f] : f \text{ is homotopic to } gp \text{ for some } g \text{ with } \mathrm{Im}(g) \subseteq U\}$. The tail-limit topology is the topology on $P^h_x(X)$ which has the collection of all $O(U, [p])$'s as a neighborhood base for $[p]$. Let $\sigma : P^h_x(X) \to X$ be the map which sends $[p]$ to the initial point of p.

Lemma 3.2. [7, Lemma 6.6] *Let X be a one-dimensional metric space and $F : [0, 1] \to P^h_x(X)$ be a path such that $F(0)$ is degenerate. If $f \in RP_x(X)$ represents $F(1)$, then $\sigma \circ F$ and f^- are homotopic.*

We remark that this statement was wrongly stated as "$\sigma \circ F$ and f are homotopic" in [7].

For a homomorphism $h : \pi_1(X, x) \to \pi_1(Y, y)$, we define $\psi : P_{X^w_h, x}(X) \to RP_y(Y)$ and $\xi : P_{X^w_h}(X) \to RP(Y)$ as follows. For a path p from $x_0 \in X^w_h$ to x, we have a reduced path q from $\tilde{h}(x_0)$ to y according to Lemma 3.1 such that the properties there hold. We define $\psi(p) = q$. For a path p_0 from $x_1 \in X^w_h$ to $x_0 \in X^w_h$ in X, p_0p is a path from x_1 to x. We define $\xi(p_0)$ to be a reduced path homotopic to $\psi(p_0p)\psi(p)^-$. We remark the following. In case $x = x_0$, $\psi(p_0)$ and $\xi(p_0)$ are defined, but these may be distinct. In particular, for the constant loop c_x on x, $\xi(c_x)$ is a degenerate path, but $\psi(c_x)$ may not.

Lemma 3.3. *Let p and p' be paths from $x_0 \in X_h^w$ to x. Then $h([p^- p']) = [\psi(p)^- \psi(p')]$. Consequently, for a loop l with base point x_0, $h \circ \varphi_p([l]) = \varphi_{\psi(p)}([\xi(l)])$.*

Proof. Let $q = \psi(p)$ and $q' = \psi(p')$. Then, as in Lemma 3.1 for each continuous $f : (\mathbb{H}, o) \to (X, x_0)$ there is a continuous map $g : (\mathbb{H}, o) \to (Y, y_0)$ such that $h \circ \varphi_{pp^-p'} \circ f_* = h \circ \varphi_{p'} \circ f_* = \varphi_{q'} \circ g_*$. Since $h \circ \varphi_{pp^-p'}(f_*([u])) = h([(p^-p')^-])h([p^-(f \circ u)p])h([p^-p'])$ for a loop $u : [0, 1] \to \mathbb{H}$ with $u(0) = u(1) = o$,

$$\begin{aligned} h([p^-p'])[q'^-][g \circ u][q']h([p^-p'])^{-1} &= h([p^-p'])\varphi_{q'}(g_*([u]))h([p^-p'])^{-1} \\ &= h([p^-p'])h \circ \varphi_{pp^-p'} \circ f_*([u])h([p^-p']^{-1}) \\ &= h([p^-(f \circ u)p]) \\ &= \varphi_q(g_*([u])) = [q^-][g \circ u][q]. \end{aligned}$$

Since q is unique up to homotopy, we have $h([p^-p'])[q'^-] = [q^-]$ and hence $h([p^-p']) = [q^-q'] = [\psi(p)^-\psi(p')]$.

Next let $p' = lp$. Then we have

$$\begin{aligned} h \circ \varphi_p([l]) &= h([p^-p']) \\ &= [\psi(p)^-\psi(lp)] \\ &= [\psi(p)^-\psi(lp)\psi(p)^-\psi(p)] \\ &= \varphi_{\psi(p)}([\xi(l)]). \end{aligned}$$

□

Lemma 3.4. *The definition of $\xi(p_0)$ does not depend on p. More precisely $\xi(p_0)$ is defined by the homotopy class $[p_0]$ of p_0 and h uniquely up to the equivalence.*

Proof. To see this let p' be another path from x_0 to x. By Lemma 3.3 $h([p^-p']) = [\psi(p)^-\psi(p')]$ and $h([(p_0p')^-(p_0p)]) = [\psi(p_0p')^-\psi(p_0p)]$. Thus

$$[\psi(p_0p')^-\psi(p_0p)] = h([(p_0p')^-(p_0p)]) = h([p'^-p]) = [\psi(p')^-\psi(p)]$$

and hence $[\psi(p_0p')^-\psi(p_0p)\psi(p)^-\psi(p')] = e$, which implies that

$$\xi(p_0) \sim \psi(p_0p)\psi(p)^- \sim \psi(p_0p')\psi(p')^-.$$

□

Lemma 3.5. *Let $x_0, x_1, x_2 \in X_h^w$ and p_0 be a path from x_1 to x_0 and p_1 be a path from x_2 to x_1 and p be a path from x_0 to x. Then $\psi(p_0p) \sim \xi(p_0)\psi(p)$ and $\xi(p_1p_0) \sim \xi(p_1)\xi(p_0)$.*

Proof. Since $\xi(p_0) \sim \psi(p_0 p)\psi(p)^-$, we have $\psi(p_0 p) \sim \xi(p_0)\psi(p)$. Now $\xi(p_1 p_0) \sim \psi(p_1 p_0 p)\psi(p)^-$ and $\xi(p_1) \sim \psi(p_1 p_0 p)\psi(p_0 p)^-$ by Lemma 3.4. Hence

$$\xi(p_1 p_0) \sim \xi(p_0)\psi(p)\psi(p)^-\xi(p_1)\psi(p_0 p)\psi(p_0 p)^- \sim \xi(p_0)\xi(p_1).$$

□

Lemma 3.6. *Let X, Y and Z be one-dimensional metric spaces and $g : \pi_1(X, x) \to \pi_1(Y, y)$ and $h : \pi_1(Y, y) \to \pi_1(Z, z)$ be homomorphisms. Let $\psi_0 : P_{X^w_g, x}(X) \to RP_y(Y)$, $\xi_0 : P_{X^w_g}(X) \to RP(Y)$, $\psi_1 : P_{Y^w_h, y}(Y) \to RP_z(Z)$, $\xi_1 : P_{Y^w_h}(Y) \to RP(Z)$ and $\psi_2 : P_{X^w_{h\circ g}, x}(X) \to RP_z(Z)$, $\xi_2 : P_{X^w_{h\circ g}}(X) \to RP(Z)$ be the maps induced from g, h and $h \circ g$ respectively. For $p \in P_{X^w_g, x}(X)$, $\psi_1(\psi_0(p)) \sim \psi_2(p)$ and for $p_0 \in P_{X^w_{h\circ g}}(X)$, $\xi_1(\xi_0(p_0)) \sim \xi_2(p_0)$.*

Proof. We remark that $X^w_{h\circ g} \subseteq X^w_g$ and that $\tilde{g}(x_0) \in Y^w_h$ for $x_0 \in X^w_{h\circ g}$.

Since $\psi_0(p)$ is determined by a continuous map $f : (\mathbb{H}, o) \to (X, x_0)$ such that $\mathrm{Im}(f_*)$ is infinitely generated instead of a continuous map from $(\mathbb{H}, o)$ to (Y, y_0), we easily get $\psi_1(\psi_0(p)) \sim \psi_2(p)$ for $p \in P_{X^w_g, x}$.

Now we have $\psi_1(\psi_0(p_0 p)) \sim \psi_2(p_0 p)$ and hence

$$\xi_1(\xi_0(p_0))\psi_1(\psi_0(p)) \sim \psi_1(\xi_0(p_0)\psi_0(p)) \sim \psi_1(\psi_0(p_0 p)) \sim \xi_2(p_0)\psi_2(p)$$

by Lemma 3.5. We have $\xi_1(\xi_0(p_0)) \sim \xi_2(p_0)$. □

The next lemma strengthens the continuity of $\tilde{h}$ on X^w_h [7, Lemma 5.3] and hence its proof is a modification of that of [7, Lemma 5.3].

Lemma 3.7. *Let X, Y be one-dimensional metric spaces, X be locally path-connected and $h : \pi_1(X, x) \to \pi_1(Y, y)$ be a homomorphism. Let $x_n \in X^w_h$ and p_n be a path from x_n to x_∞ for each $n < \omega$ such that $\mathrm{Im}(p_n)$ converge to $x_\infty \in X^w_h$. Then $\mathrm{Im}(\xi(p_n))$ converge to $\tilde{h}(x_\infty)$.*

Proof. To prove this by contradiction, suppose the negation of the conclusion. By choosing a subsequence we have a neighborhood U of $\tilde{h}(x_\infty)$ such that $\mathrm{Im}(\xi(p_n)) \not\subseteq U$. Let p be a path from x_∞ to x. According to [7, Lemma 6.1] we have an arbitrary small loop l_n with base point x_n such that $h \circ \varphi_{p_n p}([l_n])$ is represented as a reduced loop $\psi(p_n p)^- l'_n \psi(p_n p)$ for a cyclically reduced loop l'_n. Hence we have a continuous map $f : (\mathbb{H}, o) \to (X, x_\infty)$ such that $f \circ \mathbf{e}_n = p_n^- l_n p_n$. Lemma 3.1 implies that we have a path q from $\tilde{h}(x_\infty)$ to y such that $h \circ \varphi_p \circ f_* = \varphi_q \circ g_*$ for a continuous map $g : (\mathbb{H}, o) \to (Y, \tilde{h}(x_\infty))$. For a sufficiently large n, we have $\mathrm{Im}(g \circ \mathbf{e}_n) \subseteq U$. On the other hand, we have $\varphi_q \circ g_*([\mathbf{e}_n]) = h \circ \varphi_p \circ f_*([\mathbf{e}_n]) = h \circ \varphi_p([p_n^- l_n p_n]) = h \circ \varphi_{p_n p}([l_n]) = [\psi(p_n p)^- l'_n \psi(p_n p)] = [\psi(p)^- \xi(p_n)^- l'_n \xi(p_n)\psi(p)] = \varphi_q([\xi(p_n)^- l'_n \xi(p_n)])$

and hence $[g \circ \mathbf{e}_n] = [\xi(p_n)^- l'_n \xi(p_n)]$. Since $\xi(p_n)$ is a reduced path and l' is cyclically reduced, one of $\xi(p_n)^- l'$ and $l'\xi(p_n)$ is reduced. Hence the image of the reduced loop of $\xi(p_n)^- l'_n \xi(p_n)$ is not contained in U by [7, Lemma 2.6]. Now $\xi(p_n)^- l'_n \xi(p_n)$ is not homotopic to a loop in U (see the first three lines of Section 4) and we have a contradiction. □

Lemma 3.8. *Let X, Y be one-dimensional metric spaces and X be locally path-connected and path-connected, and $h : \pi_1(X, x) \to \pi_1(Y, y)$ be a homomorphism. Let $x_n, y_n \in X^w_h$ and p_n be a path from y_n to x_n for each $n < \omega$ such that $\mathrm{Im}(p_n)$ converge to $x_\infty \in X^w_h$. Then $\mathrm{Im}(\xi(p_n))$ converge to $\tilde{h}(x_\infty)$.*

Proof. By the path-connectivity and local path-connectivity, we have a path f_n from x_n to x_∞ for each n such that $\mathrm{Im}(f_n)$ converge to x_∞. To show the conclusion by contradiction, we suppose the negation. As in the proof of Lemma 3.7, by choosing a subsequence we have a neighborhood U of $\tilde{h}(x_\infty)$ such that $\mathrm{Im}(\xi(p_n)) \not\subseteq U$. Since $\xi(p_n) \sim \xi(p_n f_n f_n^-) \sim \xi(p_n f_n)\xi(f_n)^-$, $\mathrm{Im}(\xi(f_n)) \not\subseteq U$ or $\mathrm{Im}(\xi(p_n f_n)) \not\subseteq U$. We choose x_n if $\mathrm{Im}(f_n) \not\subseteq U$, and y_n otherwise. Then we have paths contradicting Lemma 3.7. □

Let X be a locally path-connected, path-connected, one-dimensional metric space, Y a one-dimensional metric space and suppose that for a homomorphism $h : \pi_1(X, x) \to \pi_1(Y, y)$, $X = X^w_h \cup \bigcup_{i \in I} A_i$, where A_is are open arcs and $A^0_i, A^1_i \in X^w_h$. Note that $\lim_{i\to\infty} \mathrm{diam}(A_i) = 0$. Then $\tilde{h} : X^w_h \to Y$ is a continuous map by [7, Lemma 5.3]. We extend $\tilde{h}$ on X as follows. For each A_i, we choose a continuous map $a_i : [0, 1] \to \overline{A_i}$ with $a_i(0) = A^0_i$ and $a_i(1) = A^1_i$ so that the restriction of a_i to $(0, 1)$ is injective. (That is, a_i is a homeomorphism, if $A^0_i \neq A^1_i$.) Then, define $\tilde{h}(x) = \xi(a_i)(a_i^{-1}(x))$ for $x \in A_i$.

Then, the continuity of $\tilde{h}$ on $\bigcup_{i \in I} A_i$ is obvious and that at a point in X^w_h follows from Lemma 3.8, since $\lim_{n\to\infty} \mathrm{diam}(A_n) = 0$. We use this extended $\tilde{h}$ in Lemmas 3.9 and 3.10 and also in the proofs of Theorems 1.1 and 1.2. In Lemma 3.10 we suppose that Y also satisfies additional properties which X has.

Lemma 3.9. *Let $h : \pi_1(X, x) \to \pi_1(Y, y)$ be a homomorphism and r be a reduced path from $x_1 \in X^w_h$ to $x_0 \in X^w_h$. Then $\xi(r)$ is homotopic to $\tilde{h} \circ r$.*

Proof. Let p_t be the restriction of r to $[1-t, 1]$ for $0 \le t \le 1$, e.g. $p_1 = r$ and p_0 is the degenerated path at x_0. Define $F : [0, 1] \to P^h_{\tilde{h}(x_0)}(Y)$ as follows. Let $F(t) = [\xi(p_t)]$, if $\sigma([p_t]) \in X^w_h$. Otherwise, we have $i \in I$ and $0 \le t_0 < t_1 \le 1$ such that $t_0 < t < t_1$, $\sigma([p_{t_0}]), \sigma([p_{t_1}]) \in X^w$

and $r \restriction [1-t_1, 1-t_0] \sim a_i$ or $r \restriction [1-t_1, 1-t_0] \sim a_i^-$. Let $F(t) = [\xi(a_i \restriction [(t_1-t)/(t_1-t_0), 1])\xi(p_{t_0})]$ if $r \restriction [1-t_1, 1-t_0] \sim a_i$ and $F(t) = [\xi(a_i^- \restriction [(t_1-t)/(t_1-t_0), 1])\xi(p_{t_0})]$ if $r \restriction [1-t_1, 1-t_0] \sim a_i^-$. If $\sigma([p_t]) \notin X_h^w$, the continuity of F at t is obvious. Otherwise, the continuity of F at t follows from Lemma 3.8. Since $F(1) = [\xi(r)]$ and $\tilde{h}(r(1-t)) = \sigma \circ F(t)$, the conclusion follows from Lemma 3.2. □

Lemma 3.10. *Let $h_0 : \pi_1(X, x) \to \pi_1(Y, y)$ be an isomorphism and h_1 be its inverse. Let p be a path between points in X^w. Then p is homotopic to $\tilde{h}_1 \circ \tilde{h}_0 \circ p$. In particular $\tilde{h}_1 \circ \tilde{h}_0(x_0) = x_0$ for $x_0 \in X^w$.*

Proof. Since a path p is homotopic to a reduced path, it suffices to show this lemma for a reduced path p. Let $\xi_0 : P_{X^w}(X) \to RP(Y)$ and $\xi_1 : P_{Y^w}(Y) \to RP(X)$ be the maps induced from h_0 and h_1 respectively. We remark that $X_{h_0}^w = X^w$ and $Y_{h_1}^w = Y^w$. Then $[\xi_1(\xi_0(p))] = [p]$ by Lemma 3.6. The conclusion follows from Lemma 3.9. □

4. Proofs of Theorems 1.1 and 1.2

The following lemma is well-known and can be proved if we notice that any path is homotopic to a reduced path in its image [3] and the reduced path is unique up to the equivalence.

Lemma 4.1. (*Folklore*) *Let X be a one-dimensional space. Then, two homotopic paths are homotopic in the union of the ranges of the two paths.*

Proof of Theorem 1.1. By Theorem 2.1 we may assume that $X = X^w \cup \bigcup_{i \in I} A_i$ and $Y = Y^w \cup \bigcup_{j \in J} B_j$, where I and J are at most countable, A_i and B_j are open arcs. Let $h_0 : \pi_1(X, x) \to \pi_1(Y, y)$ be an isomorphism and $h_1 : \pi_1(Y, y) \to \pi_1(X, x)$ be its inverse. We assume that $\tilde{h}_0$ and $\tilde{h}_1$ are extended ones on X and Y respectively according to the description before Lemma 3.9. By Lemma 3.10, a_i and $\tilde{h}_1 \circ \tilde{h}_0 \circ a_i$ are homotopic for each i. By Lemma 4.1 we have a homotopy $H_i : [0,1] \times [0,1] \to X$ such that

$$H_i(s,0) = a_i(s), \quad H_i(s,1) = \tilde{h}_1 \circ \tilde{h}_0 \circ a_i(s),$$
$$H_i(0,t) = a_i(0), \quad H_i(1,t) = a_i(1)$$

and the image of H_i is contained in $\mathrm{Im}(a_i) \cup \mathrm{Im}(\tilde{h}_1 \circ \tilde{h}_0 \circ a_i)$. Since $\lim_{i \to \infty} \mathrm{diam}(A_i) = 0$, $\lim_{i \to \infty} \mathrm{diam}(\mathrm{Im}(H_i)) = 0$. Define $\overline{H}_i : A_i \times [0,1] \to X$ by: $\overline{H}_i(x,t) = H_i(a_i^{-1}(x), t)$. Then

$$\overline{H}_i(x,0) = a_i(a_i^{-1}(x)) = x, \quad \overline{H}_i(A_i^0, t) = a_i(0) = A_i^0,$$
$$\overline{H}_i(A_i^1, t) = a_i(1) = A_i^1, \quad \overline{H}_i(x,1) = \tilde{h}_1 \circ \tilde{h}_0 \circ a_i(a_i^{-1}(x)) = \tilde{h}_1 \circ \tilde{h}_0(x).$$

Define $\overline{H} : X \times [0,1] \to X$ by: $\overline{H}(x,t) = x$ for $x \in X^w$ and $0 \leq t \leq 1$ and $\overline{H}(x,t) = \overline{H}_i(x,t)$ for $x \in A_i$ and $0 \leq t \leq 1$. Then the continuity of $\overline{H}$ follows from those of $\overline{H}_i$ s and the fact that $\lim_{i\to\infty} \mathrm{diam}(\mathrm{Im}(\overline{H}_i)) = 0$. Hence $\tilde{h}_1 \circ \tilde{h}_0$ is homotopic to the identity map on X and similarly $\tilde{h}_0 \circ \tilde{h}_1$ is homotopic to the identity map on Y. □

For our proof of Theorem 1.2 one more notion is necessary. Let $h : \pi_1(x,x) \to G$ be a homomorphism. We call a subset S of X is h-simply connected, if for every point x_0 in S, every loop l in S with base point x_0 and every path q from x_0 to x, $h(\varphi_q([l]))$ is trivial. We remark that, if $h(\varphi_q([l]))$ is trivial, then $h(\varphi_{q'}([l]))$ is also trivial for every path q' from x_0 to x.

Proof of Theorem 1.2. If X_h^w is empty, then $O_h^X = X$ and we have a brick partition $\mathcal{P}$ such that

(1) $\mathcal{P}$ is of order 2;
(2) $\overline{P}$ is h-simply connected for $P \in \mathcal{P}$;
(3) if $\overline{P} \cap \overline{Q} \neq \emptyset$, then $\overline{P} \cup \overline{Q}$ is h-simply connected for $P, Q \in \mathcal{P}$.

Hence this is the case when our procedure stops at the first step and we can easily get the conclusion from the following arguments. Hence we proceed to the case when $X_h^w \neq \emptyset$. We construct a retract $K \cup X_h^w$ of X, where K is locally homeomorphic to finite graphs in $\overline{O_h^X}$. This is a modification of the proof of Theorem 2.1 in Section 2.

Starting from the 0-step we let $\mathcal{P}_0 = \{ \text{X} \}$, but do not define A_i s and so on as before. We trace the previous proof, but we set buffers around $\bigcup\{\overline{P} : P \in \mathcal{P}_m, \overline{P} \cap X_h^w \neq \emptyset\}(= R_m)$. We call $P \in \mathcal{P}_m$ a buffer element, if $\overline{P} \cap R_m \neq \emptyset$ but $P \cap R_m = \emptyset$.

the After m-th step, we have at most finitely many points y_{mn} on the boundary of $\bigcup\{\overline{P} : P \in \mathcal{P}_m, \overline{P} \cap R_m \neq \emptyset\}$. Applying Proposition 2.2, we take a brick partition $\mathcal{P}_{m+1}$ of R_m which satisfies the following:

(1) $\mathcal{P}_{m+1}$ is of order 2 and $\mathrm{Mesh}(\mathcal{P}_{m+1}) < 1/(m+1)$;
(2) $\mathcal{P}_{m+1}$ refines the restriction of $\mathcal{P}_m$ to R_m;
(3) if $P \in \mathcal{P}_m$ is a buffer element, then $\overline{P} \cap R_m$ is a disjoint union of sets of the form $\overline{P} \cap \overline{Q}$ with $Q \in \mathcal{P}_{m+1}$ and $Q \cap R_{m+1} = \emptyset$;
(4) if $\overline{Q} \cap \overline{Q'} \neq \emptyset$, $\overline{Q} \cap X_h^w = \emptyset$ and $\overline{Q}' \cap X_h^w = \emptyset$ for $Q, Q' \in \mathcal{P}_{m+1}$, then $\overline{Q} \cup \overline{Q'}$ is h-simply connected;
(5) if $\overline{Q} \cap \overline{P} \neq \emptyset$ and $\overline{Q} \cap X_h^w = \emptyset$ for $Q \in \mathcal{P}_{m+1}$ and a buffer element $P \in \mathcal{P}_m$, then $\overline{Q} \cup \overline{P}$ is h-simply connected.

We successively construct a finite tree T_P for each buffer element $P \in \mathcal{P}_m$ such that end points of T_P are chosen from ∂P as follows. First we choose one point from each non-empty $\overline{P} \cap \overline{P'}$ for $P' \in \mathcal{P}_m$ with $P' \notin R_m$ and $P' \neq P$, making sure this point is among the points

y_{mn} whenever possible and making sure that our choice is consistent with whichever point from $\partial P'$ might have already been chosen. Then we choose one point from a non-empty $\overline{P} \cap \overline{Q}$ with $Q \in \mathcal{P}_{m+1}$. Since P is connected, we have a tree $T_P \subseteq \overline{P}$ such that $T_P \cap P$ is connected and the end points of T_P are the elements chosen from ∂P. Now each end point of T_P corresponds to some $\overline{P} \cap \overline{P'}$ or to some $\overline{P} \cap \overline{Q}$. Since T_P is an absolute extensor, we have a retraction $r_P : \overline{P} \to T_P$ so that $r_P(\overline{P} \cap \overline{P'}) = \{v\}$ or $r_P(\overline{P} \cap \overline{Q}) = \{v\}$ or for each end point $v \in T_P$ with $v \in \overline{P} \cap \overline{P'}$ or $v \in \overline{P} \cap \overline{Q}$ respectively. Next we define trees T_Q and retractions $r_Q : \overline{Q} \to T_Q$ for $Q \in \mathcal{P}_{m+1}$ with $\overline{Q} \cap R_{m+1} = \emptyset$ similarly as for $P \in \mathcal{P}_m$. We do not define T_Q nor r_Q for buffer elements $Q \in \mathcal{P}_{m+1}$ in this step. But, on a part of ∂Q for a buffer element Q retractions r_P or $r_{Q'}$ have been defined. We enumerate the retracted points of those parts of boundaries as y_{m+1n} $(n \in I_{m+1})$. More exactly, $y_{m+1\,n}$ $(n \in I_{m+1})$ is the one to one enumeration of the points in $T_P \cap \overline{Q}$ for buffer elements $P \in \mathcal{P}_m$ and buffer elements $Q \in \mathcal{P}_{m+1}$ or in $T_Q \cap \overline{Q'}$ for non-buffer elements $Q \in \mathcal{P}_{m+1}$ and buffer elements $Q' \in \mathcal{P}_{m+1}$.

For each $y_{ml} \in \overline{P}$ where $P \in \mathcal{P}_m$ is a buffer element, we have some $P' \in \mathcal{P}_m$ such that $\overline{P} \cap \overline{P'} \neq< \emptyset$ and $\overline{P'} \cap X^w \neq \emptyset$. Then we have some $y_{m+1\,n}$ in $\overline{P'}$, since $\overline{P'} \cap X^w \neq \emptyset$. We can connect y_{ml} and $y_{m+1\,n}$ by an arc in T_P and T_Q s for $Q \subseteq P'$. Using these arcs we construct D_{mn} and A_i similarly as in the proof of Theorem 2.1 and have D_n, A_i and the desired $K = \bigcup_n D_n \cup \bigcup_i A_i$. Let $X_0 = X^w_h \cup \bigcup_n D_n \cup \bigcup_i A_i$ and $r : X \to X_0$ be the retraction obtained as a union of r_P and the identity on X^w_h.

First we assume $x \in X^w_h$. We remark $r(x) = x$. To trace our previous proof, we show $h([l]) = h([r \circ l])$ for any loop l with base point x. By [10, Theorem 1], $\pi_1(Y, y)$ is a subgroup of an inverse limit of free groups. It suffices to show that $g \circ h([l^-(r \circ l)]) = e$ for every homomorphism g from $\pi_1(Y, y)$ to a free group. By [8, Theorem 1.3] (cf. [9]), we have $\varepsilon > 0$ such that any open connected subset of X of diameter less than ε is $g \circ h$-simply connected. Since $\lim_{m\to\infty} \mathrm{Mesh}(\mathcal{P}_m) = 0$, we may choose $\mathcal{P}_m$ so that $\mathrm{Mesh}(\mathcal{P}_m) < \varepsilon/2$. We have a brick partition $\mathcal{P}$ of X such that $\mathcal{P}$ consists of $P \in \mathcal{P}_m$ with $\overline{P} \cap R_m \neq \emptyset$ and $P \in \mathcal{P}_i$ with $P \cap R_i = \emptyset$.

For a given loop l with base point x, we have $0 = u_0 < u_1 < \cdots < u_k = 1$ such that $l([u_i, u_{i+1}]) \subseteq \overline{P} \cup \overline{P'}$ for $P, P' \in \mathcal{P}$. If $\overline{P} \cap \overline{P'} \neq \emptyset$ for $P \in \mathcal{P} \cap \mathcal{P}_i$ with $i < m$ and $P' \in \mathcal{P}$, then $\overline{P} \cup \overline{P'}$ is h-simply connected according to the effect of buffers. On the other hand, if $\overline{P} \cap \overline{P'} \neq \emptyset$ for $P, P' \in \mathcal{P} \cap \mathcal{P}_m$, then $\mathrm{diam}(\overline{P} \cup \overline{P'}) < \varepsilon$ and hence $\mathrm{diam}(\overline{P} \cup \overline{P'})$ is $g \circ h$-simply connected. Since each $P \in \mathcal{P}$ is path-connected, working from k

to 0 we see that $l^-(r\circ l)$ is homotopic to a concatenation of loops $l_k \cdots l_1$ such that $g\circ h([l_i]) = e$. Now we have proved that $g\circ h([l^-(r\circ l)]) = e$ and hence $h([r\circ l]) = h(l)$. Let $i : X_0 \hookrightarrow X$ be the inclusion map. Then for each loop l in X_0 with base point x we have $h\circ i_*([l]) = h([l])$ and $(X_0)^w_{h\circ i_*} = X^w_h$.

The space Z obtained from this X_0 as in the proof in Section 2 is a one-dimensional Peano continuum which is homotopy equivalent to X_0. Let $g_0 : X_0 \to Z$ and $g_1 : Z \to X_0$ be the homotopy equivalence. According to our construction we have $X^w_h \subseteq Z$ and g_0 and g_1 are the identity on X^w_h. Hence $Z^w_{h\circ i_*\circ g_{1*}} = X^w_h$. Let $h_0 = h\circ i_*\circ g_{1*}$ and $Z = Z^w_{h_0} \cup \bigcup_i A_i$ where A_i s are open arcs. We use $\widetilde{h_0}$ as the extended one on Z defined just before Lemma 3.9. Let c_x be the constant path x and $q = \psi(c_x)$, where ψ and ξ are defined for h_0. For a loop l with base point x in X, using Lemma 3.3 we have

$$\begin{aligned} h([l]) &= h\circ i_*([r\circ l]) = h\circ i_*\circ g_{1*}([g_0\circ r\circ l]) \\ &= \varphi_q([\xi(g_0\circ r\circ l)]) \\ &= \varphi_q(\widetilde{h}_{0*}([g_0\circ r\circ l])) = \varphi_q\circ(\widetilde{h_0}\circ g_0\circ r)_*([l]). \end{aligned}$$

Now $\widetilde{h_0}\circ g_0\circ r$ is the desired continuous map.

When $x \notin X^w_h$, we choose a path p from $x_0 \in X^w_h$ to x. Then we have $h\circ\varphi_p : \pi_1(X, x_0) \to \pi_1(Y, y)$. By the preceding we have a continuous map $f : X \to Y$ and a path q from $f(x_0)$ to y such that $h\circ\varphi_p = \varphi_q\circ f_*$. Let l be a loop with base point x, then

$$\begin{aligned} h([l]) &= h\circ\varphi_p\circ\varphi_{p^-}([l]) = h\circ\varphi_p([plp^-]) \\ &= \varphi_q\circ f_*([plp^-]) = \varphi_q([(f\circ p)f\circ l(f\circ p)^-]) \\ &= \varphi_q\circ\varphi_{(f\circ p)^-}\circ f_*([l]) \\ &= \varphi_{q(f\circ p)^-}\circ f_*([l]), \end{aligned}$$

which completes our proof.

Proof of Corollary 1.3. By Theorem 2.1 we may assume $X = X^w \cup \bigcup_i A_i$ where each A_i is an open arc. If $X^w = \emptyset$, then we have the conclusion easily. We suppose that $x_0 \in X^w$. Since f_* is injective, we have $X^w = X^w_{f_*}$ and we can define $(f_*)\tilde{}$ on X according to the definition just before Lemma 3.9. For $x \in X^w_{f_*}$, $(f_*)\tilde{}(x)$ is the unique point determined by Lemma 3.1 and hence we have $(f_*)\tilde{}(x) = f(x)$. Let $\psi : P_{X^w_{f_*},x_0}(X) \to RP_y(Y)$ and $\xi : \cup P_{X^w_{f_*}}(X) \to RP(Y)$ be defined as before Lemma 3.3. For $x \in A_i$, we have that $(f_*)\tilde{}(x) = \xi(a_i)(a_i^{-1}(x))$. Let p be a path from A_i^1 to x_0. Then $\xi(a_i) \sim \psi(a_i p)\psi(p)^-$. By the uniqueness of $[\psi(p)]$ according to Lemma 3.1, we have $[\psi(p)] = [f\circ p]$

and $[\psi(a_i p)] = [f \circ (a_i p)]$ by the same argument. Now, we have

$$\xi(a_i) \sim \psi(a_i p)\psi(p)^- \sim (f \circ a_i)(f \circ p)(f \circ p)^- \sim f \circ a_i$$

and so $(f_*)\tilde{} = \xi(a_i) \circ a_i^{-1}$ and the restriction of f to $\overline{A_i}$ are homotopic. By Lemma 4.1 f and $(f_*)\tilde{}$ are homotopic as in the proof of Theorem 1.1. Now Lemma 3.10 implies the conclusion. □

Remark 4.2. (1) Theorem 1.2 may hold in the case Y is a planar continuum using a method by C. Kent [12].
(2) For topologists who are not familiar with wild topology, it seems to be difficult to understand what ideas work in the proofs of this paper. Theorem 2.1 reduces Peano continua to simple ones. The idea of its proof is standard in continuum theory going back to [1]. More unfamiliar parts seem to be in Section 3, which are on an extended line of [7]. In [7] we show that many things about wild algebraic topology can be reduced to the Hawaiian earring and how the fundamental group of the Hawaiian earring works well by the non-commutative Specker phenomenon. This phenomenon goes back to G. Higman and is explained in [7, Remark 3.16(4)]. The Higman theorem is related to the fundamental group and the shape group of the Hawaiian earing. An application to topology of this theorem appeared in [5], which was used in [11]. A more apparent topological use can be seen in [6, Corollary 2.11], where it is shown that every endomorphism on the fundamental group of the Hawaiian earring is conjugate to the homomorphism induced from a continuous map. This is a prototype of Lemma 3.1, by which we define $\tilde{h}$ on X^w_h for a homomorphism h between the fundamental groups of one-dimensional Peano continua. In the present paper we have extended the domain of $\tilde{h}$ to the whole space X using Theorem 2.1 and strengthening results in [7] according to the ideas there. In other published papers topological use of the non-commutative Specker phenomenon can be seen in [2] and [16], though it is used implicitly there.

5. Acknowledgements

The author thanks G. Conner, U. Karimov, K. Kawamura and D. Repovš for their interests in this subject. The author is grateful to the referee for his very careful reading of this paper and also of [7] and his suggestions and detecting errors. The author was supported by the Grant-in-Aid for Scientific research (C) of Japan No. 20540097.

References

[1] R. H. Bing, *Partitioning continuous curves*, Bull. Amer. Math. Soc. **58** (1952), 1101–1110.

[2] J. W. Cannon and G. R. Conner, *On the fundamental groups of one dimensional spaces*, Topology Appl. **153** (2006), 2648–2672.

[3] M. L. Curtis and M. K. Fort, *The fundamental group of one-dimensional spaces*, Proc. Amer. Math. Soc. **10** (1959), 140–148.

[4] R. J. Daverman, *Decompositions of manifolds*, Academic Press, 1986.

[5] K. Eda, *Free σ-products and noncommutatively slender groups*, J. Algebra **148** (1992), 243–263.

[6] ———, *Free σ-products and fundamental groups of subspaces of the plane*, Topology Appl. **84** (1998), 283–306.

[7] ———, *The fundamental groups of one-dimensional spaces and spatial homomorphisms*, Topology Appl. **123** (2002), 479–505.

[8] ———, *Algebraic topology of Peano continua*, Topology Appl. **153** (2005), 213–226.

[9] G. R. Conner and K. Eda, *Correction to: "Algebraic topology of Peano continua" and "Fundamental groups having the whole information of spaces"*, Topology Appl. **154** (2007), 771–773.

[10] K. Eda and K. Kawamura, *The fundamental groups of one-dimensional spaces*, Topology Appl. **87** (1998), 163–172.

[11] ———, *The singular homology of the Hawaiian earring*, J. London Math. Soc. **62** (2000), 305–310.

[12] C. Kent, *Homomorphisms into the fundamental group of one-dimensional and planar continua*, 2008, Master Thesis, Brigham Young University.

[13] R. Engelking, *General topology*, Heldermann Verlag, 1989.

[14] J.C.Mayer, Lex G.Oversteengen, and E.D.Tymchatyn, *The Menger curve characterization and extension of homeomorphisms of non-locally-separating closed subsets*, PWN Dissert. Math. CCLII, 1986.

[15] M. Meilstrup, *Classifying homotopy types of one-dimensional Peano continua*, 2005, Master Thesis, Brigham Young University.

[16] A. Zastrow, *Construction of an infinitely generated group that is not a free-product of surface groups and abelian groups, but which acts freely on an $\mathbb{R}$-tree*, Proc. Roy. Soc. Edinburgh Ser. A **128** (1998), 433–445.

School of Science and Engineering, Waseda University, Tokyo 169-8555, JAPAN

E-mail address: eda@logic.info.waseda.ac.jp