Singular homology groups of one-dimensional Peano continua

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Fundamental groups and singular homology groups

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The singular homology groups H_1 are the abelianizations of the fundamental groups and consequently they possibly may be less abundant. They are not only less abundant, but are scarce, and they have the same simple classification as the Čech homology groups and shape groups.

Singular homology groups

Theorem [E4]. Let X be a one-dimensional Peano continuum. Then the singular homology group $H_1(X)$ is isomorphic to a free abelian group of finite rank or the singular homology group of the Hawaiian earring $H_1(\mathbb{H})$

$$\cong \mathbb{Z}^{\omega} \oplus \oplus_{\mathbf{c}}\mathbb{Q} \oplus \Pi_{p:\mathrm{prime}} A_p,$$

where ω is the least infinite ordinal, $\mathbf{c}$ is the cardinality of the continuum and A_p is the p-adic completion of the free abelian group of rank $\mathbf{c}$ [EK1].

A gap in my proof and filling it

A sequence of non-degenerate reduced paths $f_1, \cdots, f_{2N}$ is of 0-form, if its concatenation $f_1 \cdots f_{2N}$ is a loop and there exist pairings $\{i_k, j_k\}$ $(1 \leq k \leq N)$ of the index set $\{1, \cdots, 2N\}$ such that $f_{i_k} \equiv f_{j_k}^-$ for $1 \leq k \leq N$.
The word 0-form means that the concatenated loop represents the trivial element in the singular homology group. We remark that the empty sequence is of 0-form.
0-form Lemma: Let l_0 be a reduced loop in a one-dimensional space X. Then, $[\,l_0\,]_h = 0$ in $H_1(X)$ if and only if l_0 is a degenerate loop or there exists a 0-form $f_1, \cdots, f_{2N}$ such that $l_0 \equiv f_1 \cdots f_{2N}$.
So far there is no proof of n-form lemma. The word n-form means that the concatenated loop represents an element divisible by n in the singular homology group.

Torsionfree algebraically compact abelian groups (review)

Well-known facts:
(1)(due to Kaplansky): It is a direct sum of the divisible subgroup ($\cong \oplus_I \mathbb{Q}$) and the direct product of A_p for primes p, where A_p is the p-adic completion of a free abelian group.

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Why is this a countart part of the Specker phenomenon and how they are complementary? (arranged one - garbage)

projective (free) — injective (divisible)
domain — range (with many homomorphisms)
The Specker phenomenon: There exist only natural homomorphisms from direct products, i.e. groups with structures admitting infinite operations.

Review continued

Less-known fact (due to Dugas-Goebel): A is algebraically compact if and only if $U(A) = UU(A)$ and $A/U(A)$ is complete under $\mathbb{Z}$-adic topology, where $U(A) = \bigcap_{n\in\mathbb{N}} n!\, A$.

$$(n+1)!|a_{n+1} - a_n \ (n \in \mathbb{N}) \quad \to$$

$$\exists a_\infty(\ \ (n+1)! \,|\, a_\infty - a_n \ (n \in \mathbb{N}) \ )$$

If A is torsionfree, $U(A) = UU(A)$ holds.
If B is torsionfree and $Ker(h)$ is complete mod-U for a homomorphism $\sigma : A \to B$, then $Ker(h)$ is a pure sunbgroup of A. In addition if A is torsionfree, we have $A \cong Ker(h) \oplus Im(h)$.

No more secret, but almost unknown Fact

It is very easy to apply these to Wild Topology. If sizes of loops or maps converge to zero, we can add infinitely many meaningful ones .
For given a_n with $(n+1)! \mid a_{n+1} - a_n$, find loops b_n with

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such that b_n is of small sizes or is equal to the sum of loops of small sizes as homology classes.

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Intuitively put

$$a_\infty = \Sigma_{n=1}^{\infty}(n+1)!b_n + a_1.$$

One necessary trick here is to make the divisibility in the noncommutative stage using the lexicographical ordering on trees.
These have been used more than twenty years.

The canonical homomorphism from the singular homology to the Čech homology

Use the equivalence between the Čech homology and the Alexander homology and map the vertices of singular simplices of subdivisions.
Let $\sigma : H_1(X) \to \check{H}_1(X)$ be the canonical homomorphism (σ is surjective for every Peano continuum [EK2, Corollary 1.2]).

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But, how can we put together arbitrarily small but sprinckled cycles and how can we insure the required properties?

Global construction

An additional idea to the twenty-years old method is a construction in [CC] for highly divisible elements or a part also in [EK2] using loops filling the given space. To combine these ideas, we need to present loops rigorously.
Given homology classes b_n such that $b_n^\vee$ are trivial. Then, we can replace b_n with arbitrarily small sizes of loops. First we take a path f filling the space and attach small loops.
Let $f_{n,i}$ be loops with $\Sigma_{i=1}^{k_n}[f_{n,i}]_h = b_n$.
We attach $n+1$ copies of $f_{n,i}$ at the predecessors and consequently $(n+1)!$ copies of them. According k_n we controll the sizes of loops, i.e. chopping to k_n pieces.

Our construction is made of $(3k+2)k_n$ pieces works. See the paper for details.

Remarks

1. The compactness is essential to the algebraic compactness of $Ker(\sigma)$.
2. On the otherhand, we have $R_{\mathbb{Z}}(Ker(\sigma)) = \{0\}$ for locally path-connected metric spaces X.
3, For a Peano continuum, σ is surjective for H_1 and for an LC^n compact metric space σ is surjective for H_{n+1}. In addition $Ker(\sigma)$ is complete mod-U. It seems to be possible to analyze these more.

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