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QUESTION AND HOMOMORPHISMS ON ARCHIPELAGO GROUPS

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ABSTRACT. The classical archipelago group is a quotient group of the fundamental group of the Hawaiian earring by the normal closure of the free group of countable rank, which is denoted by $\mathcal{A}(\mathbb{Z})$. Since the fundamental group of the Hawaiian earring is expressed by the free σ-product $\times_\omega \mathbb{Z}$, we obtain an archipelago group $\mathcal{A}(G)$ by replacing $\mathbb{Z}$ with G. In [1] the authors asserted that $\mathcal{A}(\mathbb{Z})$ and $\mathcal{A}(\mathbb{Z}/k\mathbb{Z})$ are isomorphic for $k \geq 3$. We clarify a gap in their proof and show that there are surjective homomorphisms between $\mathcal{A}(\mathbb{Z}/k\mathbb{Z})$'s and $\mathcal{A}(\mathbb{Z})$ for $k \geq 2$.

1. Introduction and definitions

The main purpose of this note is to state the main question about archipelago groups and to investigate the homomorphisms defined in [1]. We also point out a gap in their proof of the main result in [1] by showing a certain property of the homomorphisms. For future developments, we define many things again and somewhat differently from [1]. Archipelago groups are the fundamental groups of so-called archipelagos, which are objects in wild algebraic topology. The reader is refered to [1] for the background.

We intend explicit presentations, but words are also used to express elements of free σ-products. For basical notions we refer to [2]. First we define archipelago groups. Let G_i $(i < \omega)$ be groups. Define $\mathcal{A}(G_i : i < \omega)$ to be the quotient group of the free σ-product $\times_{i<\omega} G_i$ factored by $N(*_{i<\omega} G_i)$, which is the normal closure of the free product $*_{i<\omega} G_i$.

Let $\sigma_G : \times_{i<\omega} G_i \to \times_{i<\omega} G_i / N(*_{i<\omega} G_i)$ and $\sigma_H : \times_{i<\omega} H_i \to \times_{i<\omega} H_i / N(*_{i<\omega} H_i)$ be the quotient homomorphisms.

Next we introduce interesting homomorphisms in [1]. Let $\varphi_i : G_i \to H_i$ for $i < \omega$ be maps which preserve the inverses, i.e. $\varphi_i(x^{-1}) = \varphi_i(x)^{-1}$. We define $\varphi : \mathcal{W}(G_i : i < \omega) \to \mathcal{W}(H_i : i < \omega)$ by: $\overline{\varphi(W)} = \{\alpha \in \overline{W} \mid \varphi_i(W(\alpha) \neq e \text{ where } W(\alpha) \in G_i\}$ and

$$\varphi(W)(\alpha) = \varphi_i(W(\alpha)), \text{ if } W(\alpha) \in G_i.$$

Then, we define $\overline{\varphi} : \times_{i<\omega} G_i \to \times_{i<\omega} H_i$ by $\overline{\varphi}(W) = \varphi(W)$ for reduced words W. Since W is restricted to reduced words, $\overline{\varphi}$ is well-defined.

Finally we define $\overline{\overline{\varphi}} : \mathcal{A}(G_i : i < \omega) \to \mathcal{A}(H_i : i < \omega)$ by: $\overline{\overline{\varphi}} \circ \sigma_G = \sigma_H \circ \overline{\varphi}$, where the well-defined-ness is assured by the fundamental homomorphism theorem.

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2. Results and proofs

A main part of the following theorem is contained in [1].

Theorem 2.1. [1] *Let φ_i be an inverse preserving map for each $i < \omega$. Then, $\overline{\overline{\varphi}}$ is a homomorphism and the non-triviality of $\overline{\overline{\varphi}}$ is equivalent to the existence of infinitely many i for which there exists an $x \in G_i$ such that $x \neq e$ and $\varphi_i(x) \neq e$.*

Proof. First we show that $\sigma_H \circ \overline{\varphi}$ is a homomorphism. Let $U, V \in \mathcal{W}(G_i : i < \omega)$ be reduced words and $W \in \mathcal{W}(G_i : i < \omega)$ be the reduced word such that $W = UV$. Then, there exists a reduced word W_0 such that

(1) $U \equiv U_0 W_0$, $V \equiv W_0^- V_0$ and $U_0 V_0$ is reduced; or
(2) $U \equiv U_0 a W_0$, $V \equiv W_0^- b V_0$ for some $a, b \in G_i$ satisfying $ab \neq e$ and $U_0(ab)V_0$ is reduced.

Therefore $W \equiv U_0 V_0$ or $W \equiv U_0(ab)V_0$ and hence $\overline{\varphi}(W) = \varphi(U_0)\varphi(V_0)$ or $\overline{\varphi}(W) = \varphi(U_0)\varphi_i(ab)\varphi(V_0)$.

Since $\varphi(W_0^-) \equiv \varphi(W_0)^-$ by preservation of the inverses,

$$\overline{\varphi}(U)\overline{\varphi}(V) = \varphi(U_0)\varphi(W_0)\varphi(W_0^-)\varphi(V_0) = \varphi(U_0)\varphi(V_0)$$

or

$$\begin{aligned} \overline{\varphi}(U)\overline{\varphi}(V) &= \varphi(U_0)\varphi_i(a)\varphi(W_0))\varphi(W_0^-)\varphi_i(b)\varphi(V_0) \\ &= \varphi(U_0)\varphi_i(a)\varphi_i(b)\varphi(V_0) \end{aligned}$$

Now, in the both bases we have

$$\sigma_H(\overline{\varphi}(U)\overline{\varphi}(V)) = \sigma_H(\varphi(U_0)\varphi(V_0)) = \sigma_H(\varphi(W))$$

and we have shown $\sigma_H \circ \overline{\varphi}$ is a homomorphism.

If there exist $x_i \in G_i$ for infinitely many i such that $x_i \neq e$ and $\varphi_i(x_i) \neq e$, the non-triviality of the map follows from considering a word obtained by ordering x_i in a natural way. Since a reduced word consists of nontrivial elements of groups G_i, the negation of the condition implies that $\varphi(W) \in *_{i<\omega} H_i$ for any reduced word $W \in \mathcal{W}(G_i : i < \omega)$, which implies $\overline{\varphi}(W) = e$. □

Since $\sigma_H \circ \overline{\varphi}(*_{i<\omega} G_i) = \{e\}$, we have a homomorphism $\overline{\overline{\varphi}} : \circledast_{i<\omega} G_i / N(*_{i<\omega} G_i) \to \circledast_{i<\omega} H_i / N(*_{i<\omega} H_i)$ such that $\sigma_H \circ \overline{\varphi} = \overline{\overline{\varphi}} \circ \sigma_G$.

An element of $\circledast_{i<\omega} G_i / N(*_{i<\omega} G_i)$ is expressed as $\sigma_G(W)$ for a word $W \in \mathcal{W}(G_i : i < \omega)$. In particular we may restrict W to be a reduced one.

Lemma 2.2. *A word W is reduced, if $W \,|\, (\alpha, \beta) \neq e$ for each pair $\alpha < \beta \in \overline{W}$ satisfying that $W(\alpha), W(\beta) \in G_{i_0}$ and no letter in G_{i_0} appears in $W \,|\, (\alpha, \beta)$ for some i_0.*

Proof. Observe that $\circledast_{i<\omega} G_i \cong G_{i_0} * \circledast_{i \neq i_0} G_i$, we see every occurrence of a letter in W remains in the reduced word of W. □

Lemma 2.3. *If $h : G \to H$ is an inverse-preserving surjective map which is not a homomorphism, then*

(1) *there exist $a, b, c \in G$ which are not the identity such that $abc \neq e$ and $h(a)h(b)h(c) = e$; or*
(2) *there exist $a, b \in G$ which are not the identity such that $ab \neq e$ and $h(a)h(b) = e$.*

Proof. In case $h(e) \neq e$, we have $a \in G$ such that $a \neq e$ and $h(a) = e$. Since $h(a^{-1}) = e^{-1} = e$, we have $a^2 = e$. Setting $b = c = a$ are desired ones for (1).

Otherwise, i.e. $h(e) = e$. Then, $h(uv) \neq h(u)h(v)$ implies $u \neq e$ and $v \neq e$ and also $uv \neq e$. Choose w so that $h(w) = h(u)h(v)$. If $w \neq e$, $a = u, b = v, c = w^{-1}$ are desired ones for (1). Otherwise, i.e. $w = e$, $a = u$ and $b = v$ are desired ones for (2). □

To define domains of words, we introduce some notions. The empty sequence is denoted by ( ) and let $n = \{0, \cdots, n-1\}$ for $n < \omega$. A finite sequence is denoted by $(i_0, \cdots, i_k)$ whose length is $k+1$. For a finite sequence $s = (i_0, \cdots, i_{k-1})$, let $s * (j) = (i_0, \cdots, i_{k-1}, j)$.

Theorem 2.4. *Suppose that $\varphi_i : G_i \to H_i$ is an inverse preserving surjective map for every $i < \omega$. If there exist infinitely many i such that φ_i are not homomorphisms, then $\overline{\overline{\varphi}}$ is never injective.*

Proof. Let J be the subset of ω consisting of all i such that φ_i are not homomorphisms. Enumerate J increasingly, i.e. $\{j_k \mid k < \omega\} = J$ and $j_k < j_{k+1}$.

Let $a_{j_k}, b_{j_k} \in G_{j_k}$ or $a_{j_k}, b_{j_k}, c_{j_k} \in G_{j_k}$ which satisfy the required properties (2) or (1) in Lemma 2.3 respectively. We define $\overline{W_\alpha} \subseteq Seq(3)$ inductively as the domain of W which is a tree with lexicographical ordering.

In the 0-step, if (2) in Lemma 2.3 holds for φ_{j_0}, then define $W((0)) = a_{j_0}, W((1)) = b_{j_0}$, and otherwise, define $W((0)) = a_{j_0}, W((1)) = b_{j_0}, W((2)) = c_{j_0}$.

Suppose that $W(s)$ is defined. Let $m = lh(s)$. As in the 0-step, if (2) in Lemma 2.3 holds for φ_{j_m}, then define $W(s * (0)) = a_{j_m}, W((s * (1)) = b_{j_m}$, and otherwise, define $W(s * (0)) = a_{j_m}, W(s * (1)) = b_{j_m}, W(s * (2)) = c_{j_m}$.

We can see that W is reduced and $\varphi(W) = e$ as follows. Since for each pair of letters indexed j_k appearing in W there appear $a_{j_{k+1}}, b_{j_{k+1}}$ between them and $a_{j_{k+1}} b_{j_{k+1}} = e$, or $a_{j_{k+1}}, b_{j_{k+1}}, c_{j_{k+1}}$ between them and $a_{j_{k+1}} b_{j_{k+1}} c_{j_{k+1}} = e$. Hence non-empty subwords of W is not equal to e. On the other hand, for every finite subset F of ω consider the projection to $*_{i \in F} H_i$ and letters indexed by the largest element j_k in F. We see $\varphi_{j_k}(a_{j_k}), \varphi_{j_k}(b_{j_k})$ or $\varphi_{j_k}(a_{j_k}), \varphi_{j_k}(b_{j_k}), \varphi_{j_k}(c_{j_k})$ appear contiguously. Since $\varphi_{j_k}(a_{j_k})\varphi_{j_k}(b_{j_k}) = e$, or $\varphi_{j_k}(a_{j_k})\varphi_{j_k}(b_{j_k})\varphi_{j_k}(c_{j_k}) = e$, we can cancel them and so on and we conclude the projectum is equal to e, which implies $\varphi(W) = e$.

Since W is a reduced word and there appear infinitely many letters, $\sigma_G(W)$ is not the identity. Since $\overline{\varphi}(W) = \varphi(W)$, $\overline{\overline{\varphi}}(\sigma_G(W)) = \sigma_H(\varphi(W)) = e$. We have shown that $\overline{\overline{\varphi}}$ is not injective.

□

Lemma 2.5. *Suppose that $\varphi_i : G_i \to H_i$ are surjective homomorphisms. Let $V \in \mathcal{W}(H_i : i < \omega)$ be a reduced word. Then, there exists a reduced word $U \in \mathcal{W}(G_i : i < \omega)$ such that $\varphi(U) \equiv V$.*

Proof. By the surjectivity of φ_i, we have $U \in \mathcal{W}(G_i : i < \omega)$ such that $\overline{U} = \overline{V}$ and $\varphi_i(U(\alpha)) = V(\alpha)$ for each $\alpha \in \overline{V}$, where $V(\alpha) \in H_i$. To show that U is reduced by contradiction, suppose that there exists a non-empty subword W of U such that $W = e$. For any $F \Subset \omega$, $W_F = e$ where W_F is a finite word such that $\overline{W_F} = \{\alpha \in \overline{W} \mid W(\alpha) \in \bigcup_{i \in F} G_i \setminus \{e\}\}$. Since φ_i is a homomorphism for each i, $\varphi(W)_F = e$, which implies V is not reduced. Now, we see that U is reduced. □

Theorem 2.6. *Suppose that $\varphi_i : G_i \to H_i$ is an inverse preserving surjective map for every $i < \omega$. Then $\overline{\overline{\varphi}}$ is surjective.*

Proof. If almost all φ_i are homomorphisms, by ignoring finitely many G_i and H_i we may assume that all φ_i are homomorphisms. Then, $\overline{\varphi}$ is surjective by Lemma 2.5. So we deal with the case that infinitely many φ_i are not homomorphisms.

For a given reduced word V, we consider $\varphi^{-1}(V)$. We cannot say it is a reduced word in $\mathcal{W}(G_i : i \in I)$ and even $\varphi^{-1}(V) \in \mathcal{W}(G_i : i \in I)$, since there may appear e in this sequence. When $V(\alpha) \in H_i$ and $\varphi_i(e) = V(\alpha)$, we replace e by letters u_i, v_i such that $u_i, v_i \neq e$ and $\varphi_(u_i)\varphi_i(v_i) = V(\alpha)$. This is done by the additional condition. Let U be the obtained one. Since such α appear only finitely many times for each i, $U \in \mathcal{W}(G_i : i \in I)$ and $\varphi(U) = V$. We claim the existence of a reduced word $U_0 \in \mathcal{W}(G_i : i \in I)$ such that $\varphi(U_0) = \varphi(U)$. Since $\varphi(U) = V$, we have $\overline{\varphi}(U_0) = V$ and hence $\overline{\overline{\varphi}}(\sigma_G(U_0)) = \sigma_H(V)$.

Actually we show the following:

> Suppose that $\varphi(U) = V$ for $U \in \mathcal{W}(G_i : i < \omega)$ and $V \in \mathcal{W}(H_i : i < \omega)$. Then, there exists a reduced word $U_0 \in \mathcal{W}(G_i : i < \omega)$ such that $\varphi(U_0) = V$.

We keep Lemma 2.2 in our mind and inserting reduced words W satisfying $\varphi(W) = e$ to U. We will define $W_\alpha \in \mathcal{W}(G_n : n \in J\}$ for each $\alpha \in \overline{U}$ such that $\varphi(W_\alpha) = e$. To state our proof rigorously we introduce some notions. Recall $3 = \{0, 1, 2\}$ and $5 = \{0, 1, 2, 3, 4\}$. We construct trees consisting of finite sequence of members of 5 whose lengths are nonzero. Enumerate $J \backslash \{0\}$ increasingly, i.e. $\{j_k \mid k < \omega\} = J \backslash \{0\}$ and $j_k < j_{k+1}$. Let $a_{j_k}, b_{j_k}, c_{j_k} \in G_{j_k}$ which satisfy the required propertied assured by Lemma 2.3.

In the first step, i.e. the 0-th step, we consider $\alpha, \beta \in \overline{U}$ such that $U(\alpha), U(\beta) \in G_0$ and $\alpha < \beta$ are contiguous, i.e. $\alpha < \gamma < \beta$ implies $U(\gamma) \notin G_0$. We admit $\beta = \infty$. We construct $W_\alpha \in \mathcal{W}(G_j : j \in J)$ similarly to W in (2), using $a, b, c \in G_j$ satisfying $abc \neq e$ and $\varphi_j(a)\varphi_j(b)\varphi_j(c) = e$. We define $\overline{W_\alpha}$ as a tree with lexicographical ordering. In the 0-substep, let u be the result of multiplications of elements of G_{j_0} appearing in the subword $U(\alpha, \beta)$ of U. We define $W_\alpha((0)) = a, W_\alpha((1)) = b, W_\alpha((2)) = c$, if $abcu \neq e$ and also $W_\alpha((3)) = a, W_\alpha((4)) = b, W_\alpha((5)) = c$ if $abcu = e$. We move β to the place of the leftmost appearance of a letter of G_{j_0} in U, if such a letter appears, and make β stay at the previous β otherwise.

Generally in the k-th substep, we let u to be the result of multiplications of letters of G appearing in $U|(\alpha, \beta)$ and define $W_\alpha(s * (0)) = a_{j_k}, W_\alpha(s * (1)) = b_{j_k}, W_\alpha(s * (2)) = c_{j_k}$ for s satisfying $lh(s) = k$. In addition if $a_{j_k} b_{j_k} c_{j_k} u = e$, we define $W_\alpha(s * (3)) = a_{j_k}, W_\alpha(s * (4)) = b_{j_k}, W_\alpha(s * (5)) = c_{j_k}$ for s which is the largest element in $\overline{W}_\alpha$ satisfying $lh(s) = k$. Then, we move β to the position of the leftmost appearance among letters whose multiplication is u in $\overline{U}$. If u does not exist, then we make β stay at the previous position. In this way we define W_α. If no letters of G_0 appear in U, we do not define anything.

Now in the m-step we consider the word obtained Y deleting all letters which do not belong to $\bigcup_{i=0}^m G_i$ from U, i.e. picking letters in $\bigcup_{i=0}^m G_i$ and order in the same way as in U. We define W_α for α satisfying $U(\alpha) \in G_m$ by letting $\beta \in \overline{U}$ to correspond to the next letter in the word in $\mathcal{W}(\bigcup_{i=0}^m G_i)$. We replace j_0 by j_m and j_k by j_{m+k}.

Our attaching W_α are done after the whole construction. Let $\overline{U_0} = \{(\alpha, s) \mid \alpha \in \overline{U}, s \in \overline{W_\alpha}$ or $s = \langle\,\rangle\}$ with the lexicographical ordering and $U_0(\alpha, \langle\,\rangle) = U(\alpha)$ and $U_0(\alpha, s) = W_\alpha(s)$ for $s \in \overline{W_\alpha}$.

The fact that $\varphi(U_0) = V$ follows from $\varphi(W_\alpha) = e$. To see that U_0 is reduced, let Y be a non-empty subword of U_0. Choose m be the least natural number such that a letter of G_m appears in Y. If there is only one letter of G_m which appears in Y, it implies $Y \neq e$. Let $\lambda, \mu \in \overline{Y}$ such that $\lambda < \mu$ and $Y(\lambda), Y(\mu)$ are contigous letters in G_m, i.e. $Y(\lambda), Y(\mu) \in G_m$ and $X(\nu) \notin G_m$ for $\lambda < \nu < \mu$.

(1) If the both appear as of form $U_0(\gamma, \langle\,\rangle)$ for some γ, then $Y(\lambda)$ and $Y(\mu)$ are considered in the m-th step. We remark that no letters of $\bigcup_{i=0}^{m-1} G_i$ appear in Y. According to considering letters in G_{j_m} in the substep 0 for W_α we conclude $Y|(\lambda, \mu) \neq e$.

(2) If $Y(\lambda)$ appears as of form $U_0(\gamma, s)$ for some γ and $s \in \overline{W_\gamma}$ and $Y(\mu)$ appears as of form $U_0(\delta, \langle\,\rangle)$ for some δ. We need to consider the remaining three cases where $Y(\lambda)$ appears as $U_0(\gamma, s)$ for some γ and $s \in W_\gamma$ and $Y(\mu)$ appears as $U_0(\delta, \langle\,\rangle)$ for some δ. There exists $k < m$ such that $U(\gamma) \in G_k$. By the minimality of m, no letter in $\bigcup_{i=0}^{m-1} G_i$ appears in Y. Hence β in the initial stage of the construction of W_γ is located to the right hand side of $Y(\mu)$. Therefore, $m = j_{k+l}$ and β in the substep l for γ is $\mu \in \overline{Y}$ and by the setting for elements of $G_{j_{k+l+1}}$ we conclude $Y(\lambda, \mu) \neq e$.

(3) If $Y(\lambda)$ appears as of form $U_0(\gamma, \langle\,\rangle)$ for some γ and $Y(\mu)$ appears as of form $U_0(\delta, s)$ for some δ and $s \in \overline{W_\delta}$. There exists $k < m$ such that $U(\delta) \in G_k$. By the minimality of m, δ is located at the left hand side of α, i.e. $\delta < \alpha$ in $\overline{U}$. Since no letters in U appear between $U_0(\delta, \langle\,\rangle)$ and $U_0(\delta, s)$, a contradiction occurs, i.e. this case does not happen.

(4) If $Y(\lambda)$ appears as of form $U_0(\gamma, s)$ for some γ and $s \in \overline{W_\gamma}$. and $Y(\mu)$ appears as of form $U_0(\delta, t)$ for some δ and $t \in \overline{W_\delta}$. By the minimality of m we have $\gamma = \delta$. Since W_γ is a reduced word $Y \mid (\alpha, \beta) \neq e$.

Now we have shown that Y is reduced. □

Corollary 2.7. *Let G_i and H_i be at most countable non-trivial groups. Then, there exists a surjective homomorphism from $\mathcal{A}(G_i : i < \omega)$ to $\mathcal{A}(H_i : i < \omega)$.*

Proof. Since $G * G'$ is inifinite for non-trivial groups G and G' and $\ast_{i<\omega}(G_{2i} * G_{2i+1}) \cong \ast_{i<\omega} G_i$, we may assume that G_i and H_i are infinite. Therefore we have an inverse-preserving map from G_i to H_i for each i and hence have the conclusion from Theorem 2.6. □

Now we have the following corollary.

Corollary 2.8. *Let G and H be groups $\mathbb{Z}$ and $\mathbb{Z}/k\mathbb{Z}$ for some $k \geq 2$.*

Remark 2.9. (1) G. Conner informed me that the surjectivity of homomorphisms in the assumption of Theorem 2.4 is essential.

(2) If there are surjections between finite groups G and H, then G and H are obviously isomorphic. There are many infinite groups for which the statement does not hold. The author debts to M. Dugas, L. Fuchs and D. Herden for this.

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