THE CLASSIFICATION OF THE INVERSE LIMITS OF SEQUENCES OF FREE GROUPS OF FINITE RANK

KATSUYA EDA AND JUN NAKAMURA

Abstract. We classify the groups which are the inverse limits of sequences of free groups of finite rank. Except free groups of finite rank there exist four groups which the inverse limits of sequences of free groups of finite rank.

Abstract. We classify the groups which are the inverse limits of sequences of free groups of finite rank. Except free groups of finite rank there exist four groups which the inverse limits of sequences of free groups of finite rank.

1. Introduction and main result

A continuum is a compact connected metric space. The first shape group of one-dimensional continua are the inverse limit of free groups of finite rank and every inverse sequence of free groups of finite rank is a pro-π_1-sequence of a one-dimensional continuum [7, p.130]. Since the first shape group is not fine enough to distinguish topologies of various one-dimensional continua, shape theorists have not paid much attention to the first shape groups. On the other hand, from group theoretic view point, it is apparently meaningful to classify the inverse limit groups of free groups of finite rank.

The present paper carries out the classification mentioned above. An inverse sequence $(G_n, p_n : n < \omega)$ is a sequence $(G_n : n < \omega)$ of groups together with a sequence of homomorphisms $p_n : G_{n+1} \to G_n$, where n runs through all non-negative integers. The inverse limit $\varprojlim(G_n, p_n : n < \omega)$ is a subgroup of the direct product $\{x \in \Pi_{n<\omega} G_n : p_n(x(n+1)) = x(n) \text{ for } n < \omega\}$ and is often denoted by G_∞. Let $\mathbb{Z}_n$ be a copy of the integer group $\mathbb{Z}$ for $n < \omega$. For groups G_0 and G_1, $G_0 * G_1$ is the free product of G_0 and G_1.

2010 *Mathematics Subject Classification.* 20E99 (primary), 55Q07, 20E05 (secondary).

Key words and phrases. inverse limit, free σ-product, Specker phenomenon, non-commutative, algebraic topology.

Theorem 1.1. *The inverse limit of any inverse sequence of free groups of finite rank is isomorphic to one of the following groups (1)-(5). Conversely, each of the groups (1)-(5) is isomorphic to the inverse limit of some inverse sequence of free groups of finite rank. Moreover, groups (1)-(5) are mutually non-isomorphic.*

(1) *free groups of rank;*
(2) $\varprojlim(G_n, p_n : n < \omega)$ *where* $G_n = *_{i<n}\mathbb{Z}_i$ *and* $p_n : G_{n+1} \to G_n$ *is the projection such that* $p_n \mid *_{i<n} \mathbb{Z}_i = \mathrm{id}$ *and* $p_n(\mathbb{Z}_n) = \{e\}$*;*
(3) *the free group of countable rank* F_ω*;*
(4) $\varprojlim(G_n, p_n : n < \omega)$ *where* $G_0 = F_\omega$, $G_{n+1} = G_n * \mathbb{Z}_n$ *and* $p_n : G_{n+1} \to G_n$ *is the projection such that* $p_n \mid G_n = \mathrm{id}$ *and* $p_n(\mathbb{Z}_n) = \{e\}$*;*
(5) $\varprojlim(G_n, p_n : n < \omega)$ *where* $G_0 = F_\omega$ *and* $G_{n+1} = G_n * F_{\omega n}$ *where* $F_{\omega n}$ *is a copy of* F_ω, $p_n : G_{n+1} \to G_n$ *is the projection such that* $p_n \mid G_n = \mathrm{id}$ *and* $p_n(F_{\omega n}) = \{e\}$.

Actually our proof classifies the limits of inverse sequence of free groups of at most countable rank.

Corollary 1.2. *For any sequence of free groups of at most countable rank, there exists a sequence of free groups of finite rank such that the inverse limits of these sequences are isomorphic.*

If the all bonding maps p_n are surjective, then by a well-known theorem [6, Proposition 2.12], the limit is isomorphic to either of the groups (1) or (2) of Theorem 1.1. Such groups are the fundamental groups of one-dimensional Peano continua.

A subgroup R of a group G is a *retract* of G, if there exists a homomorphism $r : G \to R$ such that $r(x) = x$ for $x \in R$. A subgroup is a free retract of G, if it is a free group which is a retract of G. Undefined notions are standard can be seen in [5].

2. Proofs

The proof of Theorem 1.1 is divided into three steps: (a) every inverse limit of an inverse sequence of free groups of finite rank is isomorphic to one of the groups listed in Theorem 1.1, (b) every group listed in Theorem 1.1 is realized as the inverse limit of a sequence of free groups of finite rank, (c) groups (1)-(5) in Theorem 1.1 are mutually non-isomorphic.

For an inverse sequence $(G_n, p_n : n < \omega)$, let $G_\infty = \varprojlim(G_n, p_n : n < \omega)$ and $q_n : G_\infty \to G_n$ the projection. Further let $q_{nm} = p_n \cdots p_{m-1}$ for $n < m$ and $H_n = q_n(G_\infty)$. We adopt the same notaion for $(G^i_n, p^i_n : n < \omega)$ for $i = 0, 1$.

Lemma 2.1. *For an inverse sequence of groups* $(G_n, p_n : n < \omega)$, $(H_n, p_n|H_{n+1} : n < \omega)$ *is an inverse sequence with surjective bonding maps and* $\varprojlim(H_n, p_n|H_{n+1} : n < \omega) = G_\infty$.

Lemma 2.2. (*A consequence of* [3, Theorem 6.4]). *Let* F_0 *and* F_1 *be free groups and* $h : F_0 \to F_1$ *be a surjective homomorphism. Then, there exists an injective homomorphism* $\sigma : F_1 \to F_0$ *and a subgroup* F_2 *of* F_0 *such that* $F_0 = F_2 * \sigma(F_1)$ *and* $h \circ \sigma = \mathrm{id}$ *and* $h(F_2) = \{e\}$.

Now we prove the statement (a):

Lemma 2.3. *Let* $(G_n, p_n : n < \omega)$ *be an inverse sequence of free groups of finite rank. Then* G_∞ *is isomorphic to one of the groups listed in Theorem 1.1.*

Proof. Since any subgroup of a free group of finite rank is a free group of at most countable rank, we assume that every p_n is surjective by Lemma 2.1 and every G_n is a free group of at most countable rank.

First we assume that infinitely many G_n are of finite rank. By taking a subsequence we may also assume that all G_n are of finite rank. If almost all p_n are injective, then the inverse limit is a free group of finite rank. If infinitely many p_n fail to be injective, we may assume that all p_n are non-injective. By Lemma 2.2 we have an injective homomorphism $\sigma_n : G_n \to G_{n+1}$ and a non-trivial free subgroup of finite rank K_n such that $G_{n+1} = K_n * \sigma_n(G_n)$ and $p_n \circ \sigma_n$ is the identity on G_n. Now $(G_n, p_n : n < \omega)$ is isomorphic to a subsequence of $(*_{i<n}\mathbb{Z}_i, p_n : n < \omega)$ in (2) of Theorem 1.1.

Next we assume that almost all G_n are of countable rank. Then we may assume that all G_n are of countable rank. If almost all p_n are injective and hence isomorphisms, then the limit is isomorphic to a free group of countable rank. Otherwise we may assume that all p_n fail to be injective. As is the argument in the first paragraph, we apply Lemma 2.2 to obtain, for each $n < \omega$, an injective homomorphism $\sigma_n : G_n \to G_{n+1}$ and a non-trivial free subgroup of at most countable rank K_n such that $G_{n+1} = K_n * \sigma_n(G_n)$ and $p_n \circ \sigma_n$ is the identity on G_n. If K_n are of finite rank for almost all n, then we see that the limit is isomorphic to the group (4). Otherwise, infinitely many K_n are of countable rank, then we conclude, by taking an appropriate subsequence, that the inverse limit is isomorphic to the group (5). □

Next we proceed to a proof of (b). A straightforward proof of the following lemma is provided for completeness.

Lemma 2.4. *For an inverse sequence of groups* $(G^i_n, p^i_n : n < \omega)$, *let* $(G_n, p_n : n < \omega)$ *be the inverse sequence defined by* $G_n = G^0_n * G^1_n$ *and* $p_n \,|\, G^i_{n+1} = p^i_n$. *Then we have* $H_n = H^0_n * H^1_n$.

Proof. Let $(x_n : n < \omega)$ be an element of G_∞ and fix an arbitrary n. Let $W_n \equiv w_{n,0} \cdots w_{n,k_n}$ be the reduced word of x_n as an element of the free product $G_n = G_n^0 * G_n^1$. Note that $w_{n,i} \in G_n^j$ if and only if $w_{n,i+1} \in G_n^{1-j}$. The desired conclusion is equivalent to $w_{n,i} \in H_n^0 \cup H_n^1$. For this purpose, we define subwords of $U_{m,n,i}$ of W_m for $m \geq n$ inductively. Let $U_{n,n,i}$ be the word of one letter $w_{n,i}$ For $i = 0, \cdots, k_n$, let $U_{n+1,n,i}$ be a subword of W_{n+1} so that $p_n(U_{n+1,n,i}) = w_{n,i}$ and $U_{n+1,n,0} \cdots U_{n+1,n,k_n} \equiv W_{n+1}$.

Suppose that $U_{m,n,i}$ is defined for $0 \leq i \leq k_n$. Then we have $U_{m,n,i} \equiv w_{m,j_0} w_{m,j_0+1} \cdots w_{m,j_1}$. Let $U_{m,n,i}$ be the word of the concatenation $U_{m+1,m,j_0} U_{m,j_0+1} \cdots U_{m+1,m,j_1}$. Then we have $p_m(U_{m+1,n,i}) = U_{m,n,i}$ and $U_{m+1,n,0} \cdots U_{m+1,n,k_n} \equiv W_{m+1}$.

Let $r_n^j : G_n^0 * G_n^1 \to G_n^j$ be the projection. If $w_{n,i} \in G_n^j$,

$$p_m(r_{m+1}^j(U_{m+1,n,i})) = r_m^j(p_m(U_{m+1,n,i})) = r_m^j(U_{m,n,i})$$

for $m \geq n$. This means $(r_m^j(U_{m,n,i}) : n < \omega)$ is an element of G_∞^j which projects $w_{n,i}$, which implies that $w_{n,i} \equiv U_{n,n,i}$ belongs to H_n^j. □

We remark that G_∞ is not isomorphic to $G_\infty^0 * G_\infty^1$ except trivial cases, e.g. when all G_n^0 are trivial groups or when all p_n^0 and p_n^1 are injective.

Let $\mathbb{Z}_{i,j}$ be copies of the integer group $\mathbb{Z}$. It is easy to construct the sequences for (1) and (2). Below we construct sequences only for (3),(4) and (5).

Lemma 2.5. *There exist inverse sequences $(G_n, p_n : n < \omega)$ of free groups of finite rank whose inverse limits are isomorphic to those of (3), (4) and (5) respectively.*

Proof. Since the commutator subgroup of the free group of rank two is a free group of countable rank, we have an injective homomorphism h_n^i from $*_{j=2n-1}^{2n+2} \mathbb{Z}_{i,j}$ to the commutator subgroup of $\mathbb{Z}_{i,2n-1} * \mathbb{Z}_{i,2n}$. Define $g_n^i : *_{j=1}^{2(n+1)} \mathbb{Z}_{i,j} \to *_{j=1}^{2n} \mathbb{Z}_{i,j}$ by: $g_n^i \mid *_{j=1}^{2(n-1)} \mathbb{Z}_{i,j}$ is the identity on $*_{j=1}^{2(n-1)} \mathbb{Z}_{i,j}$ and $g_n^i \mid *_{j=2n-1}^{2n+2} \mathbb{Z}_{i,j} = h_n^i$ for $1 \leq n < \omega$. We remark that g_0^i maps $\mathbb{Z}_{i,1} * \mathbb{Z}_{i,2}$ to $\{e\}$ and g_n^i are injective for $n > 0$.

In order to realize the group (3), let $G_n = *_{j=1}^{2n} \mathbb{Z}_{0,j}$ and $p_n = g_n^0$ for $1 \leq n < \omega$. Since each p_n is injective, $G_\infty \cong \bigcap_{m>1} q_{1m}(G_m)$. Since $\bigcap_{m>1} q_{1m}(G_m)$ is a subgroup of G_1, $\bigcap_{m>1} q_{1m}(G_m)$ is a free group of at most countable rank. It suffices to show that it is not finitely generated. Since $\bigcap_{m>1} q_{1m}(G_m) \cong *_{j=1}^{2n} \mathbb{Z}_{0,j} * \bigcap_{m>n} q_{nm}(*_{j=2n+1}^{2m} \mathbb{Z}_{0,j})$, $\bigcap_{m>1} q_{1m}(G_m)$ has a free retract of $2n$ rank and hence is not finitely generated.

Next we proceed to realize the group(4). Let $G_n = *_{j=1}^{2n}\mathbb{Z}_{0,j} * *_{j=1}^{n}\mathbb{Z}_{1,j}$ and let $p_n | *_{j=1}^{2(n+1)} \mathbb{Z}_{0,j} = g_n^0$ and $p_n | *_{j=1}^{n} \mathbb{Z}_{1,j}$ to be the identity on $*_{j=1}^{n}\mathbb{Z}_{1,j}$ and $p_n(\mathbb{Z}_{1,n+1}) = \{e\}$. We apply Lemma 2.4 for $G_n^0 = *_{j=1}^{2n}\mathbb{Z}_{0,j}$ and $G_n^1 = *_{j=1}^{n}\mathbb{Z}_{1,j}$. Obviously $H_n^1 = *_{j=1}^{n}\mathbb{Z}_{1,j}$ and, since g_n^0 is injective for $n > 0$, we have $H_n^0 = \bigcap_{m>n} q_{nm}^0(*_{j=1}^{2m}\mathbb{Z}_{0,j})$. Here $\bigcap_{m>n} q_{nm}^0(*_{j=1}^{2m}\mathbb{Z}_{0,j})$ is a free group of countable rank by the above proof for (3) and hence this inverse sequence, i.e. $(H_n, p_n|H_n : 1 \leq n < \omega)$ is the sequence of (4) in Theorem 1.1.

Finally for the group (5), let $G_n = *_{i=0}^{n} *_{j=1}^{2(n-i+1)}\mathbb{Z}_{i,j}$ for $1 \leq n < \omega$ and let $p_n : G_{n+1} \to G_n$ be a homomorphism such that $p_n | *_{j=1}^{2(n-i+2)} \mathbb{Z}_{i,j} = g_{n+1-i}^i$ for $0 \leq i \leq n$. We remark $p_n(\mathbb{Z}_{n+1,1} * \mathbb{Z}_{n+1,2}) = g_0^{n+1}(\mathbb{Z}_{n+1,1} * \mathbb{Z}_{n+1,2}) = \{e\}$. We need to analyze H_n. We shall show that $H_n = *_{i=0}^{n} S_n^i$, where $S_n^i \leq *_{j=1}^{2(n-i+1)}\mathbb{Z}_{i,j}$ and S_n^i is a free group of countable rank for $n \geq i$, and $p_n|S_{n+1}^i$ is an isomorphism for $n \geq i$, which implies the conclusion.

To show this for S_n^0, we apply Lemma 2.4 for $G_n^0 = *_{j=1}^{2(n+1)}\mathbb{Z}_{0,j}$ and $G_n^1 = *_{i=1}^{n} *_{j=1}^{2(n-i+1)} \mathbb{Z}_{i,j}$. As in the proof of (4), we see that $H_n^0 = S_n^0$ is a free group of countable rank and $p_n|S_{n+1}^0 : S_{n+1}^0 \to S_n^0$ is an isomorphism for $n \geq 0$. For S_n^1 we apply Lemma 2.4 to the inverse sequence $G_n^1 = *_{i=1}^{n} *_{j=1}^{2(n-i+1)} \mathbb{Z}_{i,j}$ instead of G_n, i.e. we consider the new $G_n^0 = *_{j=1}^{2n}\mathbb{Z}_{1,j}$ and the new $G_n^1 = *_{i=2}^{n} *_{j=1}^{2(n-i+1)} \mathbb{Z}_{i,j}$, and conclude that $p_n|S_{n+1}^1 : S_{n+1}^1 \to S_n^1$ is an isomorphism for $n \geq 1$ and we see S_n^1 is a free group of countable rank. Consequently, we have $H_1 = S_1^0 * S_1^1$. For S_n^k, we work inductively on $*_{i=k}^{n} *_{j=1}^{2(n-i+1)} \mathbb{Z}_{i,j}$ instead of G_n and conclude S_n^k is a free group of countable rank and $p_n|S_{n+1}^k \to S_n^k$ is an isomorpism for $n \geq k$. □

Next we recall a generalization of the Higman theorem.

Lemma 2.6. [2, Theorem 1.2]*(Weak form)*
Let $(G_n, p_n : n < \omega)$ is an inverse sequence such that each p_n is surjective. For any homomorphism h from G_∞ to a free group F, there exists an $m < \omega$ and a homomorphism $\overline{h} : G_m \to F$ such that $h = \overline{h} \circ p_m$.

Lemma 2.7. *Let $(G_n, p_n : n < \omega)$ be an inverse sequence. If R_n is a retract of G_n and $r_n : G_n \to R_n$ is a retraction for each n such that $p_n \circ r_{n+1} = r_n \circ p_n$, then $(R_n, p_n|R_n : n < \omega)$ is a inverse sequence and $\varprojlim(R_n, p_n|R_n : n < \omega) = R_\infty$ is a retract of G_∞.*

Proof. Define $r(x)(n) = r_n(x(n))$ for $x \in R_\infty$. Then we have

$$p(r(x)(n+1)) = p_n \circ r_{n+1}(x(n+1)) = r_n \circ p_n(x(n+1)) = r_n(x(n)) = r(x)(n)$$

and hence $r(x) \in R_\infty$ and $r(x) = x$ for $x \in R_\infty$. $\square$

Applying this lemma, we have

Lemma 2.8. *Let $(G_n, p_n : n < \omega)$ be an inverse sequence such that $G_n = *_{i=0}^n H_i$ and $p_n | *_{i=0}^n H_i$ is the identity and $p_n(H_{n+1}) = \{e\}$. Then the subgroup*

$$\overline{G_n} = \{x \in G_\infty \,|\, x(k) = x(n) \textit{ for } k \geq n, x(k) = q_{kn}(x(n)) \textit{ for } k < n\}$$

is a retract of G_∞ and isomorphic to G_n.

We identify $\overline{G_n}$ with G_n and simply write G_n. Also we specify $\rho_n : G_\infty \to G_n$ to be the retraction as above.

By Lemmas 2.3 and 2.5, it suffices to show the non-isomorphicness and particularly to show the non-isomorphicness among three uncountable groups (2), (4), (5). The groups (4) and (5) have a free retract of countable rank G_0 by Lemma 2.8. Now we show (2) has no free retract of countable rank. Suppose that $r : G_\infty \to R$ be a retraction to a free retract R of countable rank. Then, by Lemma 2.6 we have an m and a homomorphism $h : G_m \to R$ such that $r = h \circ \rho_m$. Since G_m is finitely generated, R is finitely generated, a contradiction.

Now what remains to be shown is the group (4) is not isomorphic to the group (5). A proof requires an involved argument is carried out in the next section.

Remark 2.9. *As in* [2, Theorem 1.2], *Lemma 2.6 also holds when F is the fundamental group of the Hawaiian earring.*

3. The non-isomorphicness of the groups (4) and (5)

For a group G let $Ab(G)$ be the abelianization of G, i.e. $Ab(G) = G/G'$. For a homomorphism $h : G_0 \to G_1$, let $Ab(h) : Ab(G_0) \to Ab(G_1)$ be the induced homomorphism.

An abelian group A is *complete mod-U*, if for a given sequence $a_n (1 \leq n < \omega)$ of elements of A satisfying $(n+1)! \,|\, a_{n+1} - a_n$ for every $1 \leq n < \omega$ there exists a_∞ such that $n! \,|\, a_\infty - a_n$ for every $1 \leq n < \omega$.

Since $\mathbb{Z}$ is not complete mod-U and the homomorphic image of a complete mod-U abelian group is again complete mod-U, we have

Lemma 3.1. [1, Proposition 4.3] *Let A be a complete mod-U abelian group. Then* $\mathrm{Hom}(A, \mathbb{Z}) = \{0\}$.

Let $G_n = *_{i=0}^n H_i$ and $p_n : G_{n+1} \to G_n$ be the projection such that $p_n \,|\, G_n = \mathrm{id}$ and $p_n(H_{n+1}) = \{e\}$.

Let $\rho_n : G_\infty \to G_n$ be the projections and also let $r_n : G_n \to H_n$ be the projections. Define $\sigma : G_\infty \to \Pi_{n<\omega} Ab(H_n)$ by $\sigma(x)(n) = Ab(r_n(\rho_n(x)))$ for $n < \omega$. The following lemma is a variant of [1, Theorem 4.7] for inverse limits.

Lemma 3.2. *The group* $\mathrm{Ker}(\sigma)/G'$ *is complete mod-U.*

Proof. We present xG' by $[x]$. If $x \in G'$, then $\rho_n(x) \in G'_n$ and $x \in \mathrm{Ker}(\sigma)$, i.e. we have $G' \leq \mathrm{Ker}(\sigma)$. Let $x \in \mathrm{Ker}(\sigma)$ and $n < \omega$. Since $(*_{i=0}^n H_i)'$ naturally becomes a subgroup of G'_∞ by Lemma 2.8, we have $y \in \mathrm{Ker}(\sigma)$ such that $[y] = [x]$ and $\rho_n(y) = e$.

Suppose that $(n+1)! \,|\, [x_{n+1}] - [x_n]$ for $1 \leq n < \omega$. We have y_n such that $y_1 = x_1$, $(n+1)![y_{n+1}] = [x_{n+1}] - [x_n]$ and $\rho_n(y_{n+1}) = e$.

The above is rewritten as $[x_n] = \Sigma_{i=1}^n i![y_i]$and hence the desired element v woulf be formally as $[v] = \Sigma_{i=1}^\infty i![y_i]$ but the limit procedure should be carried out carefully so that $(n+1)! \,|\, [v] - [x_n]$. In order to make as appropreate procedure, we use a tree with lexicographical ordering to express elements in a non-commutative group.

Let Seq be the set of all finite sequences of natural numbers and denote the length of $s \in Seq$ by $lh(s)$, i.e. $s = \langle s(1), \cdots, s(lh(s)) \rangle$. The empty sequence has length 0. For $s, t \in Seq$, $s \prec t$ if $s(i) < t(i)$ for the minimal i with $s(i) \neq t(i)$ or t extends s properly.

Let $D_{m,n} = \{s \in Seq : 0 \leq lh(s) \leq n, 1 \leq s(i) \leq i + m \text{ for } 1 \leq i \leq n\}$ and $W_{m,n} : D_{m,n} \to G_n$ with the ordering $\prec$ and $W_{m,n}(s) = \rho_n(y_{m+lh(s)})$. Then, under the ordering $\prec$, $W_{m,n}$ express an element of G_n, e.g.

$$\begin{aligned} W_{1,2} &\equiv \rho_2(y_1)\rho_2(y_2)\rho_2(y_3)\rho_2(y_3)\rho_2(y_3)\rho_2(y_2)\rho_2(y_3)\rho_2(y_3)\rho_2(y_3) \\ &= \rho_2(y_1)\rho_2(y_2)\rho_2(y_2). \end{aligned}$$

Then, it is easy to see $p_n(W_{m,n+1}) = W_{m,n}$ and hence we have $g_m \in G_\infty$ such that $\rho_n(g_m) = W_{m,n}$.

We observe $g_m = y_m g_{m+1}^{m+1}$. Hence we have

$$[g_1] = \Sigma_{i=1}^n i![y_i] + (n+1)![g_{n+1}]$$

and consequently $(n+1)! \,|\, [g_1] - [x_n]$. □

Lemma 3.3. [4, Theorem 94.5] *Let* $h : \mathbb{Z}^\omega \to \oplus_I \mathbb{Z}$ *be a homomorphism. Then there exists* $n < \omega$ *and a homomorphism* $\overline{h} : \mathbb{Z}^n \to \oplus_I \mathbb{Z}$ *such that* $h = \overline{h} \circ \rho_n$*, where* $\rho_n : \mathbb{Z}^\omega \to \mathbb{Z}^n$ *is the projection.*

Lemma 3.4. *The abelian group* $\Pi_{i<\omega}(\oplus_{j<\omega}\mathbb{Z}_{i,j})$ *is a homomorphic image of the group of (5), but is not a homomorphic image of the group of (4).*

Proof. Let $(G_n, p_n : n < \omega)$ be the inverse sequence of (5). Then, $Ab(p_n) : Ab(G_{n+1}) \to Ab(G_n)$ is a homomorphism from $(\oplus_\omega \mathbb{Z})^{n+1}$ to $(\oplus_\omega \mathbb{Z})^n$ such that the restriction of $Ab(p_n)$ to $(\oplus_\omega \mathbb{Z})^n$ is the identity and $Ab(p_n)$ maps the last copy of $\oplus_\omega \mathbb{Z}$ to $\{0\}$. Hence $\varprojlim(Ab(G_n), Ab(p_n) : n < \omega)$ is isomorphic to $\Pi_{i<\omega}(\oplus_{j<\omega}\mathbb{Z}_{i,j})$ and hence $\Pi_{i<\omega}(\oplus_{j<\omega}\mathbb{Z}_{i,j})$ is a homomorphic image of the group of (5).

Next h be a homomorphism from G_∞ to $\Pi_{i<\omega}(\oplus_{j<\omega}\mathbb{Z}_{i,j})$, where $(G_n, p_n : n < \omega)$ is the inverse sequence of (4). Since the range is a subgroup of a direct product of copies of $\mathbb{Z}$, the restriction of h to $\mathrm{Ker}(\sigma)$ is the zero homomorphism by Lemmas 3.2 and 3.1. Hence we have a homomorphism $\overline{h} : G_\infty / \mathrm{Ker}(\sigma) \to \Pi_{i<\omega}(\oplus_{j<\omega}\mathbb{Z}_{i,j})$. Since $\varprojlim(G_n, p_n : n < \omega)/\mathrm{Ker}(\sigma) \cong \oplus_\omega \mathbb{Z} \oplus \mathbb{Z}^\omega$, we may assume that $\overline{h}$ is a homomorphism from $\oplus_\omega \mathbb{Z} \oplus \mathbb{Z}^\omega$ to $\Pi_{i<\omega}(\oplus_{j<\omega}\mathbb{Z}_{i,j})$. Let $r_i : \Pi_{i<\omega}(\oplus_{j<\omega}\mathbb{Z}_{i,j}) \to \oplus_{j<\omega}\mathbb{Z}_{i,j}$ be the projection for each i. By Lemma 3.3 we have $k_i < \omega$ such that $r_i \circ \overline{h}(\mathbb{Z}^\omega) \leq \oplus_{j<k_i}\mathbb{Z}_{i,j}$. Let $r : \Pi_{i<\omega}(\oplus_{j<\omega}\mathbb{Z}_{i,j}) \to \Pi_{i<\omega}(\oplus_{j\geq k_i}\mathbb{Z}_{i,j})$ be the projection. Then we have $r \circ \overline{h}(\mathbb{Z}^\omega) = \{0\}$. Since $r \circ \overline{h}(\oplus_\omega \mathbb{Z})$ is at most countable, we conclude that $\overline{h}$ is not surjective and consequently h is not surjective. $\square$

Since Lemma 3.4 implies that the group of (4) is not isomorphic to that of (5), we have completed our proof of Theorem 1.1.

Acknowledgements The authurs thank to the referee for detecting errors in the first version and helps for the exposition and wordings.

References

[1] K. Eda, *Free σ-products and noncommutatively slender groups*, J. Algebra **148** (1992), 243–263.
[2] ———, *The non-commutative Specker phenomenon*, J. Algebra **204** (1998), 95–107.
[3] H. Federer and B. Jonsson, *Some properties of free groups*, Trans. Amer. Math. Soc. **68** (1950), 1–27.
[4] L. Fuchs, *Infinite abelian groups, vol. 1,2*, Academic Press, 1970,1973.
[5] M. Hall Jr., *The theory of groups*, Macmillan, 1959.
[6] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer, 1977.
[7] S. Mardešić and J. Segal, *Shape theory* (*the inverse system approach*), North Holland, 1982.

School of Science and Engineering, Waseda University, Tokyo 169, JAPAN
E-mail address: eda@logic.info.waseda.ac.jp