SINGULAR HOMOLOGY GROUPS OF ONE-DIMENSIONAL PEANO CONTINUA

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ABSTRACT. Let X be a one-dimensional Peano continuum. Then the singular homology group $H_1(X)$ is isomorphic to a free abelian group of finite rank or the singular homology group of the Hawaiian earring.

1. INTRODUCTION AND MAIN RESULT

The study of singular homology of one-dimensional spaces is back to Curtis and Fort [3]. They proved that for every one-dimensional separable metric space X the singular homology groups $H_k(X) = \{0\}$ for $k \geq 2$.

A Peano continuum is a locally connected, connected, compact metric space. As we have proved previously, the fundamental groups of one-dimensional Peano continua determine their homotopy types [8], and in particular the fundamental groups of one-dimensional Peano continua which are not semi-locally simply connected everywhere determine their homeomorphism types [7]. Consequently, the fundamental groups of one-dimensional Peano continua are abundant. We recall that the Hawaiian earring is the plane compactum

$$\mathbb{H} = \{(x, y) : (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2} : 1 \leq n < \omega\}.$$

It is known that the singular homology group of the Hawaiian earring is isomorphic to the abelian group

$$\mathbb{Z}^\omega \oplus \oplus_{\mathbf{c}}\mathbb{Q} \oplus \Pi_{p:\text{prime}} A_p,$$

where ω is the least infinite ordinal, $\mathbf{c}$ is the cardinality of the continuum and A_p is the p-adic completion of the free abelian group of rank $\mathbf{c}$ [11, Theorem 3.1] (see Remark 1.3).

In contrast to the case of the fundamental groups, we have

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Theorem 1.1. *Let X be a one-dimensional Peano continuum. Then the singular homology group $H_1(X)$ is isomorphic to a free abelian group of finite rank or the singular homology group of the Hawaiian earring .*

The proof shows,

Corollary 1.2. *Let X be a one-dimensional Peano continuum. If X is locally semi-simply connected, then $H_1(X)$ is isomorphic to a free abelian group of finite rank. Otherwise, $H_1(X)$ is isomorphic to the singular homology group of the Hawaiian earring .*

The result is somewhat unexpected, because the classification is the same as those of the Čech homology groups and and the shape groups (Čech homotopy groups) of one-dimensional Peano continua, while that of the fundamental groups is different, which we have mentioned above. Though proofs for the classifications of the Čech homology groups and the shape groups are done rather geometrically, the proof for the singular homology groups is highly group theoretic as we show in the sequel.

As well-known, M. G. Barratt and J. Milnor [1] proved that the three dimensional singular homology group of the two dimensional Hawaiian earring is non-trivial, which shows a counter-intuitive behavior of singular homology. Our result is another counter-intuitive one even in the dimension one.

Remark 1.3. The proof of [11, Theorem 3.1] depends on [6, Lemma 4.11]. However there is a gap in the proof of [6, Lemma 4.11]. Hence we prove Lemma 3.6 in the present paper and trace another way of proofs and generalize [11, Theorem 2.1].

2. Sequences and abelian groups

To express finite or infinite sequences of paths and elements of groups, we introduce some notion, which we have used in [6, 5, 9]. Let Seq be the set of all finite sequences of non-negative integers and denote the length of $s \in Seq$ by $lh(s)$. The empty sequence is denoted by $(\ )$. For $s, t \in Seq$, let $s * t$ be the concatenation of s and t, i.e. $lh(s*t) = lh(s)+lh(t)$ and $(s*t)_i = s_i$ for $1 \leq i \leq lh(s)$ and $(s*t)_i = t_{i-lh(s)}$ for $lh(s)+1 \leq i \leq lh(s)+lh(t)$. Generally $s \in Seq$ is denoted by $(s_1, \cdots, s_n)$ where $s_k (1 \leq k \leq n)$ are non-negative integers and $n = lh(s)$. The lexicographical ordering is denoted by $\preceq$, i.e. for $s, t \in Seq$, $s \preceq t$ if $s_i < t_i$ for the minimal i with $s_i \neq t_i$ or t extends s. For a non-empty sequence $s \in Seq$, let $s^+ \in Seq$ be the sequence such that $lh(s^+) = lh(s)$ and $s^+_i = s_i$ for $i < lh(s)$ and $s^+_i = s_i + 1$ for $i = lh(s)$.

We summarize notions for abelian groups. Hence in this section a group means an abelian group. For a group A, the *Ulm subgroup* $U(A)$ of A is $\bigcap\{n!A : n < \omega\}$. If A is torsionfree, $U(A)$ becomes to be the divisible subgroup $D(A)$ of A. The divisible subgroup is a direct summand of A. A torsionfree divisible group is the direct sum of copies of the rational group $\mathbb{Q}$.

A group A is called *complete mod-U*, if $A/U(A)$ is complete [16, VII 39], i.e. for a given $a_n \in A(n \in \mathbb{N})$ such that $n! \,|\, a_{n+1} - a_n$, there exists an element a such that $n! \,|\, a - a_n$ for every $n \in \mathbb{N}$.

It is known that a group A is *algebraically compact*, if and only if A is complete mod-U and $U(U(A)) = U(A)$ [4]. If A is torsionfree, then $U(A) = U(U(A)) = D(A)$. Hence, a torsionfree, complete mod-U group is algebraically compact. The structure of a torsionfree algebraically compact group is well-known and determined by cardinalities depending on primes [16, p.169]. Let $\widehat{\mathbb{Z}}$ be the $\mathbb{Z}$-completion of $\mathbb{Z}$ [16, p. 164]. Then $\widehat{\mathbb{Z}} \cong \Pi_{p:prime}\mathbb{J}_p$, where $\mathbb{J}_p$ is the p-adic integer group.

A subgroup S of a group A is *pure*, if, for $a \in S$, $n \,|\, a$ in A implies $n \,|\, a$ in S. It is known that a group A is algebraically compact, if and only if A is *pure-injective*, i.e. if A is a pure subgroup of a group B, then A is a direct summand of B.

For a group A, $R_{\mathbb{Z}}(A)$ is the subgroup $\bigcap\{\mathrm{Ker}(h) : h \in \mathrm{Hom}(A, \mathbb{Z})\}$, which is a radical, i.e. $R_{\mathbb{Z}}(A/R_{\mathbb{Z}}(A)) = \{0\}$. It is easy to see that $A/R_{\mathbb{Z}}(A)$ is a subgroup of the direct product of copies of the integer group $\mathbb{Z}$. For undefined notions for abelian groups, we refer the reader to [16].

3. Paths in one-dimensional metric spaces and group theoretic properties

To investigate the divisibility in $H_1(X)$ we recall reduced paths on the line of thinking in [7].

For $a \leq b$, a continuous map $f : [a, b] \to X$ is called a path from $f(a)$ to $f(b)$. The points $f(a)$ and $f(b)$ are called the initial point and the terminal point of f respectively. When $a = b$, the path f is said to be *degenerate*. A loop f is a path with $f(a) = f(b)$. For a path $f : [a, b] \to X$, f^- denotes a path such that $f^-(s) = f(a + b - s)$ for $a \leq s \leq b$. Two paths $f : [a, b] \to X, g : [c, d] \to X$ are *equivalent*, denoted by $f \equiv g$, if there exists a homeomorphism $\varphi : [a, b] \to [c, d]$ such that $\varphi(a) = c, \varphi(b) = d$ and $f = g \cdot \varphi$. Two paths $f : [a, b] \to X$ and $g : [c, d] \to X$ are *homotopic* if there exists a continuous map H whose domain is the quadrangle in the plane with the vertexes $(a,0),(b,0),(c,1)$ and $(d,1)$ such that

$$\begin{cases} H(s,0)=f(s) & \text{for } a\le s\le b,\\ H(s,1)=g(s) & \text{for } c\le s\le d,\\ H((1-t)a+tc,t)=f(a)=g(c) & \text{for } 0\le t\le 1,\\ H((1-t)b+td,t)=f(b)=g(d) & \text{for } 0\le t\le 1.\end{cases}$$

The homotopy class containing a path f is denoted by $[f]$. The homotopy defined above is usually called "a homotopy relative to end points." Since the homotopies that appear in this paper have this property, we drop the term "relative to end points" for simplicity.

A path $f:[a,b]\to X$ is *reduced* if each subloop of f is not null-homotopic, that is, for each pair $u<v$ with $f(u)=f(v)$, $f\restriction[u,v]$ is not null-homotopic. Note that a constant map is reduced if and only if it is degenerate. For paths $f:[a,b]\to X$ and $g:[c,d]\to X$ with $f(b)=g(c)$, fg denotes the concatenation of f and g, that is, a path from $[a,b+d-c]$ to X such that $fg(s)=f(s)$ for $a\le s\le b$ and $fg(s)=g(s-b+c)$ for $b\le s\le b+d-c$. A loop f is *cyclically reduced* if ff is reduced. An *arc* A between points x and y is a subspace of X which is homeomorphic to the unit interval $[0,1]$ whose end points are x and y.

Lemma 3.1. [7, Lemma 2.4] *Let X be a one-dimensional normal space. Then every path is homotopic to a reduced path and the reduced path is unique up to equivalence.*

Lemma 3.2. [7, Lemma 2.5] *For a reduced loop f, there exist a unique reduced path g and a unique reduced loop h up to equivalence such that $f\equiv g^-hg$ and h is cyclically reduced.*

Lemma 3.3. [7, Lemma 2.6] *Let X be a one-dimensional space. For reduced paths $f:[a,b]\to X$ and $g:[c,d]\to X$ with $f(b)=g(c)$, there exist unique paths h, f' and g' up to equivalence such that*

- *$f\equiv f'h^-$ and $g\equiv hg'$;*
- *$f'g'$ is a reduced path.*

Though any path in a one-dimensional space X is homotopic to a reduced path (Lemma 3.1), there's no effective reduction steps in general (see Example 3.9). However, if $f_1f_2\cdots f_n$ is a path in X and each f_i is a reduced path, we have the reduced path of $f_1f_2\cdots f_n$ by cancellations using Lemma 3.3 at most $n-1$-times, i.e. we have a finite step reduction. For a loop f in a space we denote the homotopy class of f by $[f]$ and the singular homology class of f by $[f]_h$.

Definition 3.4. A sequence of non-degenerate reduced paths $f_1, \cdots, f_{2N}$ is of *0-form*, if its concatenation $f_1 \cdots f_{2N}$ is a loop and there exist pairings $\{i_k, j_k\}$ $(1 \le k \le m)$ of the index set $\{1, \cdots, 2N\}$ such that $f_{i_k} \equiv f_{j_k}^-$ for $1 \le k \le N$.

The word 0-form means that the concatenated loop represents the trivial element in the singular homology group. We remark that the empty sequence is of 0-form.

Definition 3.5. The *length* of a 0-form $f_1, \cdots, f_{2N}$ is N and its *rank* is the cardinality of the set $\{i : f_i f_{i+1}$ is not reduced for $1 \le i \le 2N-1\}$.

Lemma 3.6. *Let l_0 be a reduced loop in a one-dimensional space X. Then, $[\, l_0 \,]_h = 0$ in $H_1(X)$ if and only if l_0 is a degenerate loop or there exists a 0-form $f_1, \cdots, f_{2N}$ such that $l_0 \equiv f_1 \cdots f_{2N}$.*

Proof. The if-part is clear and we show the other direction. Since any loop is homotopic to a unique reduced loop up to the equivalence by Lemma 3.1 and the homopotopy class of a 0-homologous loop belongs to the commutator subgroup of the fundamental group by the Poincaré-Hurewicz theorem, it suffices to show that any 0-homologous loop is homotopic to a reduced loop of 0-form.

We prove the lemma by induction on the rank r and the length N where the ordering of pairs (r, N) is lexicographical. We remark this ordering is a wellordering, which assures our induction works. If $r = 0$, then the loop of 0-form is reduced and we have the conclusion. On the other hand if $N = 1$, then $f_1 f_2$ is homotopic to a degenerate loop. Hence we proceed to the induction steps.

We introduce a *basic reduction* of a 0-form $f_1, \cdots, f_{2N_0}$. Suppose that $f_{i+1} \cdots f_{2N_0}$ is reduced and $f_i \cdots f_{2N_0}$ is not reduced. Let r_0 be the rank of $f_1, \cdots, f_{2N_0}$. By Lemma 3.3 we have $f_i \equiv f_i' h, f_{i+1} \cdots f_{2N_0} \equiv h^- f_{i+1}'$ such that $f_i' f_{i+1}'$ is reduced. A basic reduction of $f_1, \cdots, f_{2N_0}$ is the following 0-form $f_1^*, \cdots f_{2N_1}^*$.

(Case 1) f_i' and f_{i+1}' are not empty: We cancel hh^-, replace f_i and f_{i+1} by f_i' and f_{i+1}' respectively and get a 0-form $f_1, \cdots f_{i-1}, f_i', f_{i+1}', f_{i+2}, \cdots, f_{2N_0}$ as $f_1^*, \cdots, f_{2N_1}^*$, whose rank is $r_0 - 1$ and $N_1 = N_0 + 1$.

(Case 2) f_i' or f_{i+1}' is empty:

(Subcase 2.1) f_i' is empty and $f_{i-1} f_{i+1}'$ is reduced, or f_{i+1}' is empty and $f_i' f_{i+2}$ is reduced:

We cancel hh^-, rearrange pairings if necessary and get a 0-form $f_1^*, \cdots, f_{2N_1}^*$. Then, in the former case $N_1 = N_0 - 1$ or the rank is $r_0 - 1$ according to the

emptiness of f'_{i+1} and in the latter case $N_1 = N_0 - 1$ or the rank is $r_0 - 1$ according to that of f'_i.
(Subcase 2.2) Otherwise, i.e. f'_i is empty and $f_{i-1}f'_{i+1}$ is not reduced, or f'_{i+1} is empty and $f'_i f_{i+2}$ is not reduced:

We get a 0-form $f^*_1, \cdots, f^*_{2N_1}$ as in Case 2.1, whose rank is equal to or less than r_0 and $N_1 = N_0$ (actually we can conclude that the rank is r_0 but it is not necessary for our argument).

Starting from a given loop l of 0-form, we iterate basic reductions. If the cases other than Subcase 2.2 appear we have the conclusion by induction hypothesis. Hence we show that Subcase 2.2 never continue infinitely many times, which completes our proof of Lemma 3.6. To the contradiction, suppose that Subcase 2.2 iterates infinitely many times starting from a loop l of 0-form. Then we have an infinite sequence of 0-forms σ_n and $0 < a_{n+1} < a_n < \cdots < a_1 = b_1 < \cdots < b_n < b_{n+1} < 1$ such that

(1) the rank and the length of σ_n are the same as those of σ_0;
(2) $(l \restriction [0, a_n])(l \restriction [b_n, 1])$ is the concatenation of paths in σ_n.

We remark $(l \restriction [a_n, a_1])^- \equiv l \restriction [b_1, b_n])$. Let $a_\infty = \inf\{a_n : n < \infty\}$ and $b_\infty = \sup\{b_n : n < \infty\}$.

In the m_0-step we have N-pairings. If the two intervals of a pair are in $[0, a_\infty] \cup [b_\infty, 1]$, then this pair is not changed in any m-step for $m \geq m_0$. For intervals appearing in some steps, we call an interval *outside*, if it is contained in $[0, a_\infty] \cup [b_\infty, 1]$ and *inside* if it is contained in $[a_\infty, b_\infty]$. We call an interval $[c, d]$ *overlapping*, if $c < a_\infty < d < b_\infty$ or $a_\infty < c < b_\infty < d$. First we claim that an outside interval never be paired with an overlapping one.

To see this by contradiction suppose an outside interval $[c_0, d_0]$ is paired with an overlapping interval $[c_1, d_1]$. We assume $c_1 < a_\infty < d_1$, since the other case is symmetric. Once $[c_0, d_0]$ and $[c_1, d_1]$ are paired, infinitely many $[u, d_0]$ are paired with some overlapping $[c_1, v]$ in some steps. This implies that there are more than N pairs appear in some step one of whose pairs are subintervals of $[c_0, d_0]$, which is a contradiction.

Next we show that after some steps outside intervals are paired with other outside intervals. If an outside intervals I is paired with an inside interval, then according to disppearing of the inside intervals I is possibly partitioned. But such partitionings for I occur only finitely many times, since this procedure fixes the number N_0 of the pairs. Now we observe a non-degenerate subinterval I_0 of I, which will not be partitioned. We claim that I_0 will be paired with an outside interval. Otherwise, I_0 is paired with infinitely many inside intervals, which implies that I_0 is the degenerate path

$l(a_\infty) = l(b_\infty)$, a contradiction. Hence we conclude that after some steps every outside interval is paired with another outside one.

We remark that if an overlapping interval does not appear in some step, then it does not appear in further steps and if an overlapping interval is paired with another overlapping interval in some step, then in further steps two overlapping intervals are paired. Next we show that after some steps overlapping intervals are paired with other overlapping intervals. To see this by contradiction, suppose that an overlapping interval $[c_0, d_0]$ with $c_0 < a_\infty < d_0 < b_\infty$ is paired with an inside interval and in further steps its overlapping subintervals are paired with inside intervals. Then as in the case of outside intervals there appear only finitely many subintervals of $[c_0, a_\infty]$ in the further steps and hence we have an overlapping interval $[c_1, d_1]$ with $c_0 \leq c_1 < a_\infty < d_1 < d_0$ such that in the further steps an overlapping interval containing a_∞ is of form $[c_1, d]$ for some $d \leq d_1$. Since $l|[c_1, a_\infty]$ is not degenerate, we have a contradiction as in the case of outside intervals. The case $a_\infty < c_0 < b_\infty < d_0$ is symmetric and we omit its proof.

These imply that after some steps every inside interval is paired with another inside interval. Now choose two points u_0, u_1 from an inside interval so that $l(u_1) \neq l(u_2)$. Then we have copies of them in some inside interval at any further steps and we have a contradiction $l(u_1) = l(a_\infty) = l(b_\infty) = l(u_2)$.

Now we have completed proof of Lemma 3.6. We remark our proof implies that the basic reductions stop in a finite step, since Subcase 2.2 never occurs infinitely many times and other cases decrease the order of a pair (r, N). $\square$

A family $\mathcal{U}$ of open subsets of a space X is *of order* 2, if $U \cap V \cap W = \emptyset$ for each distinct $U, V, W \in \mathcal{U}$. If a space X is one-dimensional, then every finite open cover has a refinement of order 2 [15].

There is a natural homomorphism from the singular homology to the Čech homology. Though we'll use a result of [12] in principle, we need to investigate the homomorphism more precisely and we present a direct presentation of the homomorphism according to [10].

For a loop l in a one-dimensional space X, we define a loop $f_\mathcal{U}$ in the nerve $X_\mathcal{U}$ as follows [14].

We take a sequence $0 = t_0 < t_1 < \cdots < t_n = 1$ and elements $U_0, \cdots, U_n \in \mathcal{U}$ with the following properties:

- ∘ $l(t_i) \in U_i$ for each $0 \leq i \leq n$ and $U_0 = U_n = x_\mathcal{U}$;
- ∘ $l([t_i, t_{i+1}]) \subset U_i \cup U_{i+1}$ for $0 \leq i < m$.

Define $l_\mathcal{U} : [0, 1] \to X_\mathcal{U}$ as $l_\mathcal{U}(t_i) = U_i$ and extend linearly on each $[t_i, t_{i+1}]$. Then, such an $l_\mathcal{U}$ is unique up to homotopy, i.e.

(1) Take another sequence $0 = t'_0 < t'_1 < \cdots < t'_n = 1$ and elements $U'_1, \cdots, U'_n \in \mathcal{U}$ and define a loop $l'_{\mathcal{U}}$ in $X_{\mathcal{U}}$ so as to satisfy the above two conditions. Then, $l_{\mathcal{U}}$ and $l'_{\mathcal{U}}$ are homotopic.

(2) If m is a loop in X homotopic to l, then $m_{\mathcal{U}}$ and $l_{\mathcal{U}}$ are also homotopic.

The natural homomorphism $\sigma : H_1(X) \to \check{H}_1(X)$ for a path-connected space X is defined by: $\rho_{\mathcal{U}}(\sigma([\,l\,]_h)) = [l_{\mathcal{U}}]_h$, where $\rho_{\mathcal{U}}$ is the projection from $\check{H}_1(X)$ to $H_1(X_{\mathcal{U}})$, and $[\,l\,]_h$ is the homology class containing l and $[l_{\mathcal{U}}]_h$ the homology class containing $l_{\mathcal{U}}$ respectively.

For the following construction we suppose that X is a locally path-connected metric space and $\mathcal{U}$ be an open cover of X consisting of path-connected sets is of order 2. Since we use this for locally path-connected spaces, we always use covers consisting of path-connected sets.

We use the preceding notation for a loop l in X and a cover of X. Let $\mathcal{U}_0 = \{U_i : 0 \leq i \leq n\} \subseteq \mathcal{U}$ be a finite cover of $\mathrm{Im}(l)$ and $p_{U_0} = l(0)$. Choose $p_U \in U$ for $U \in \mathcal{U}_0$ with $U \neq U_0$. Then, using the path-connectivity of U and V we inductively define an arc $A_{UV} = A_{VU} \subseteq U \cup V$ between p_U and p_V for $U, V \in \mathcal{U}_0$ with $U \cap V \neq \emptyset$ so that A_{UV} is the unique arc between p_U and p_V in $(U \cup V) \cap \bigcup\{A_{UV} : U, V \in \mathcal{U}_0\}$. Then $\bigcup\{A_{UV} : U, V \in \mathcal{U}_0\}$ is homeomorphic to a finite graph and $(U \cup V) \cap \bigcup\{A_{UV} : U, V \in \mathcal{U}_0\}$ is simply-connected for each $U, V \in \mathcal{U}_0$. We remark that p_U may not be a branching point in this finite graph and A_{UU} is the one point set $\{p_U\}$. Since $\mathcal{U}$ is infinite, to avoid a tedious argument, we do not construct a graph in X for the nerve $X_{\mathcal{U}}$.

Next we construct a loop $\bar{l}$ in the finite graph $\bigcup\{A_{UV} : U, V \in \mathcal{U}\}$ for a loop l with base point U_0 in the nerve $X_{\mathcal{U}_0}$, which is a finite graph, so that a path in the edge UV corresponds to a path from p_U to p_V on the arc A_{UV}.

Then we apply this construction to the above loop $l_{\mathcal{U}}$. Then $\overline{l_{\mathcal{U}}} \restriction [t_i, t_{i+1}]$ is a path from p_{U_i} to $p_{U_{i+1}}$ on the arc $A_{U_i U_{i+1}}$ and $\overline{l_{\mathcal{U}}}(0) = l(0) = l(1) = l_{\mathcal{U}}(1)$.

Lemma 3.7. *Let X be a one-dimensional locally path-connected metric space. If l is a loop such that $[\,l\,]_h \in \mathrm{Ker}(\sigma)$, then l is homologous to the sum of arbitrary small cycles. In addition, arbitrary small cycles can be chosen in the image of l.*

Proof. Let l be a loop with $[\,l\,]_h \in \mathrm{Ker}(\sigma)$. For a given cover $\mathcal{V}$, according to the paracompactness of X we have a locally finite refinement $\mathcal{V}_0$ of $\mathcal{V}$. By Dowker's theorem [15, 7.2.4], we have an open 2-cover $\mathcal{V}_1$ which refines $\mathcal{V}_0$. Let $\mathcal{U}$ be the set of all path-connected components of some $V \in \mathcal{V}_1$. Then $\mathcal{U}$ is a 2-cover consisting of path-connected open sets. Hence, for a given $\varepsilon > 0$

we can choose an open 2-cover $\mathcal{U}$ of X which consists of path-connected open sets with size less than $\varepsilon/2$. Taking sufficiently large n, according to the preceding construction we have $0 = t_0 < t_1 < \cdots < t_n = 1$, $U_i \in \mathcal{U}$, $\mathcal{U}_0$, p_U for $U \in \mathcal{U}_0$, $l_{\mathcal{U}}$ and $\overline{l_{\mathcal{U}}}$.

Let q_i be a path from p_{U_i} to $l(t_i)$. Since $[l_{\mathcal{U}}]_h = 0$, we have a partition of the index set $\{0, 1, \cdots, n-1\} = \{i_k, j_k : 1 \leq k \leq m\} \cup S$ such that $n = 2m + |S|$ and $l \restriction [t_{j_k}, t_{j_k+1}] = (l \restriction [t_{i_k}, t_{i_k+1}])^-$ and $U_i = U_{i+1}$ for each $i \in S$. We remark that this is the edge-path version of the 0-form in Lemma 3.6. Then, $\overline{l_{\mathcal{U}}}$ is a null-homologous loop in X. We have

$$
\begin{aligned}
& [\, l\,]_h - [\,\overline{l_{\mathcal{U}}}\,]_h \\
= \;& [\, l\,]_h - [\,\overline{l_{\mathcal{U}}}\,]_h + \Sigma_{i=1}^{n-1}[q_i (q_i)^-]_h \\
= \;& [(l \restriction [t_0, t_1])\, q_1\, (\overline{l_{\mathcal{U}}} \restriction [t_0, t_1])^-]_h \\
& \quad + \Sigma_{i=2}^{n-2}[(l \restriction [t_i, t_{i+1}])\, q_{i+1} (\overline{l_{\mathcal{U}}} \restriction [t_i, t_{i+1}])^- (q_i)^-]_h \\
& \quad + [(l \restriction [t_{n-1}, t_n])(\overline{l_{\mathcal{U}}} \restriction [t_{n-1}, t_n])^- q_n^-]_h.
\end{aligned}
$$

Since the homology classes of cycles in the last summations are of size less than ε and $[\,\overline{l_{\mathcal{U}}}\,]_h = 0$, we have the conclusion.

For the additional statement, we remark that $\mathrm{Im}(l)$ is a Peano continuum and every path in X is homotopic to the reduced path in its image. Thus, the preceding proof can be done in $\mathrm{Im}(l)$ and we have the additional statement. □

Lemma 3.8. *Let X be a one-dimensional locally path-connected metric space. Then $R_{\mathbb{Z}}(H_1(X)) \leq \mathrm{Ker}(\sigma)$ holds.*

Proof. Decompose X to the path-connected components X_i ($i \in I$). Then we have $H_1(X) = \oplus_{i \in I} H_1(X_i)$ and $R_{\mathbb{Z}}(H_1(X)) = \oplus_{i \in I} R_{\mathbb{Z}}(H_1(X_i))$. Hence, without loss of generality we assume that X is path-connected. To prove $R_{\mathbb{Z}}(H_1(X)) \leq \mathrm{Ker}(\sigma)$ by contradiction, suppose that $\sigma([\, l\,]_h) \neq 0$ and $[\, l\,]_h \in R_{\mathbb{Z}}(H_1(X))$ for a loop l. According to the fact in the proof of Lemma 3.7, we have a 2-cover $\mathcal{U}$ consisting of path-connected open sets such that $0 \neq [\, l_{\mathcal{U}}\,]_h \in H_1(X_{\mathcal{U}})$. Since $H_1(X_{\mathcal{U}})$ is a free abelian group, we conclude $[\, l\,]_h \notin R_{\mathbb{Z}}(H_1(X))$, which is a contradiction. □

Example 3.9. We show the existence of a loop l which is homotopic to the constant loop, but does not contain a non-degenerate subloop of form ff^-. We denote the clockwise winding to the i-th circle of the Hawaiian earring $\mathbb{H}$ by a_i. Let $Seq(2)$ be the subset of Seq consisting of sequences of $0, 1$. We define a loop as an infinite concatenation of loops whose sizes converge to zero. Let $\overline{l} = Seq(2) \setminus \{(\,)\}$ and l be the loop obtained by concatenating a_i

and a_i^- according to the lexicographical ordering of $\bar{l}$, i.e.

$$l \restriction [\Sigma_{i=1}^{n-1} 2^{-2i} + \Sigma_{i=1}^{n} s(i) 2^{-2i+1}, \Sigma_{i=1}^{n-1} 2^{-2i} + \Sigma_{i=1}^{n} s(i) 2^{-2i+1} + 2^{-2n}] \equiv a_i$$

if $s_n = 0$ and

$$l \restriction [\Sigma_{i=1}^{n-1} 2^{-2i} + \Sigma_{i=1}^{n} s(i) 2^{-2i+1}, \Sigma_{i=1}^{n-1} 2^{-2i} + \Sigma_{i=1}^{n} s(i) 2^{-2i+1} + 2^{-2n}] \equiv a_i^-$$

if $s_n = 1$, where $n = lh(s)$.

To show that l is homotopic the constant loop, let p_n be the projection of $\mathbb{H}$ to the bouquet B_n consisting of the first n circles. Then, $p_n \circ l$ is a loop in B_n and it is easy to see that $p_n \circ l$ is null-homotopic. Then l itself is null-homotopic [10, Thm 1]. The reason of the non-existence of a subloop of l of form ff^- follows from the fact that in l each a_i and a_i^- have immediate successors, but have no immediate predecessor.

The next example shows that we cannot replace the notion of the reducedness of a loop in a space X with a sequence of reduced loops in the nerves of X.

Example 3.10. We construct a reduced loop l in $\mathbb{H}$ such that each projection of l to B_n is not reduced for $1 \leq n < \omega$. The construction is similar to the above. Let $\bar{l} = Seq(2) \setminus \{\langle\ \rangle\}$ and concatenating $a_i a_i$ and a_i^- according to the the lexicographical ordering on $\bar{l}$ instead of concatenating a_i and a_i^-.

The fact that $p_n \circ l$ is not reduced can be seen as follows. Consider the appearance of $a_n a_n$ in $p_n \circ l$. Then, a_n^- follows immediately, i.e. there is a subloop $a_n a_n a_n^-$ of $p_n \circ l$ and hence $p_n \circ l$ is not reduced. To see the reducedness of l by contradiction suppose that a non-degenerate subloop l' of l is null-homotopic. Without loss of generality we may assume that the base point of l' is o. Then l' should be an infinite concatenation of a_i. Let n be the minimal number such that a_n or a_n^- appears in l'. Since l' is null-homotopic, the times of appearances of a_n and a_n^- are the same. In the subloop between neighboring a_n and a_n^-, or a_n^- and a_n, a_{n+1} appears one time more than a_{n+1}^- and hence l' is not null-homotopic. Hence, l is reduced.

4. Construction of loops

For our construction of loops and cycles we prepare some notions which have been used in [6, 5, 9], but some modification is necessary, since we need to treat with loops with different base points. Though such a treatment has been done by J. Cannon and G. Conner in the proof of [2, Theorem 6.7], their presentation is not sufficiently precise to prove the next lemma. To prove it an exact presentation on the line as that we have done in the previous section is preferable, and we follow the line in [6, 5, 9].

Suppose that natural numbers k_i are given. Let $S = \{s \in Seq : 0 \leq s_i < k_i$ for $1 \leq i \leq lh(s)\}$ and for $s \in S$ let $a_s = \Sigma_{i=1}^{lh(s)} s_i / \Pi_{j=1}^{i} k_j$. Next let $T = \{t \in Seq : 0 \leq t_i < (i+1)k_i$ for $1 \leq i \leq lh(t)\}$. Let $S_m = \{s \in S : lh(s) = m\}$ and $T_m = \{t \in T : lh(t) = m\}$. For $t \in Seq$ with $0 \leq t_i < (i+1)k_i$, define $s_t, c_t \in Seq$ with $lh(s_t) = lh(c_t) = lh(t)$ by:

$$(i+1)(s_t)_i + (c_t)_i = t_i, \quad 0 \leq (s_t)_i < k_i, \quad 0 \leq (c_t)_i < i+1.$$

Let

$$\begin{aligned} b_t &= \Sigma_{i=1}^{lh(t)}((3i+2)(s_t)_i + (c_t)_i + 1)/\Pi_{j=1}^{i}(3j+2)k_j \\ &= \Sigma_{i=1}^{lh(t)}(3t_i - (s_t)_i + 1)/\Pi_{j=1}^{i}(3j+2)k_j \end{aligned}$$

and $\varepsilon_m = 1/\Pi_{i=1}^{m}(3i+2)k_i$. If $(c_t)_{lh(t)} < lh(t) = m$ for $t \in T$, then we have $t^+ \in T$ and $b_{t^+} = b_t + 3\varepsilon_m$. But, if $(c_t)_{lh(t)} = lh(t) = m$, then $b_t + 3\varepsilon_m$ is not equal to any $b_{t'}$ for $t' \in T$. We remark that $a_s \leq a_{s'}$ if and only if $s \preceq s'$ for $s, s' \in S$ and $b_t \leq b_{t'}$ if and only if $t \preceq t'$ for $t, t' \in T$.

Let $f : [0,1] \to X$ be a path.

> (*) Suppose that we are given finite open covers $\mathcal{U}_n$ of $\mathrm{Im}(f)$ such that each $U \in \mathcal{U}_n$ is path-connected, the diameter of each $U \in \mathcal{U}_n$ is less than $1/n$, and $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$, and also suppose that $U_s \in \mathcal{U}_{lh(s)}$ and k_n are chosen as $f([a_s, a_{s^+}]) \subseteq U_s$ and $U_t \subseteq U_s$ for $s \prec t$.

Let l_s be a loop in $U_s \in \mathcal{U}_{lh(s)}$ with the base point $f(a_s)$ for $s \in S$ with $lh(s) = n$. Let $\alpha_{m+1} = \Sigma_{i=1}^{m} \Sigma_{s \in S_i} (i+1)!\, [l_s]_h + \alpha_1$ in $H_1(X)$ for $m \geq 1$. Our purpose is to define a path g along f so that $g \cdot f^-$ is a loop and $(m+1)! \mid [g \cdot f^-]_h + \alpha_1 - \alpha_m$ for each $m \in \mathbb{N}$.

For $t \in T_m$, define $g \restriction [b_t, b_t - \varepsilon_m] \equiv l_{s_t}$ and for $t \in T$ with $lh(t) = m$ and $0 \leq (c_t)_m < m$, define $g \restriction [b_t + \varepsilon_m, b_t + 2\varepsilon_m] \equiv (f \restriction [a_{s_t}, a_{s_t^+}])^-$. If we define these for $t \in T$ for $lh(t) \leq m$, the parts in $[0,1]$ where we have not defined are $\bigcup_{t \in T_m}(b_t, b_t + \varepsilon_m) \cup \{1\}$. For t satisfying $t_i = (i+1)(k_i - 1) + i$ (for $1 \leq i \leq m = lh(t)$), we have $b_t + \varepsilon_m = 1$. If $g(x)$ is defined for $x \in (b_t, b_t + \varepsilon_m)$, then $g(x) \in U_{s_t}$. Hence g uniquely extends to a continuous map on $[0,1]$, which we also denote by g. Now g is a path from $f(0)$ to $f(1)$ and hence gf^- is a loop. We'll show that

$$[gf^-]_h - \Sigma_{i=1}^{m-1} \Sigma_{s \in S_i} (i+1)!\, [l_s]_h$$

is divided by $(m+1)!$.

For a fixed $1 \leq m < \omega$, we cut g into finitely many pieces and consider an element of the chain group:

$$\begin{aligned} \Sigma_{i=1}^{m-1}\Sigma_{t\in T_i}\, g\restriction [b_t-\varepsilon_i, b_t] \quad &+ \quad \Sigma_{i=1}^{m-1}\Sigma_{t\in T_i, 0\leq (c_t)_i<i}\, g\restriction [b_t+\varepsilon_i, b_t+2\varepsilon_i] \\ &+ \quad \Sigma_{t\in T_m}\, g\restriction [b_t, b_t+\varepsilon_m]. \end{aligned}$$

We see that $g\restriction [b_t-\varepsilon_i, b_t] \equiv l_{s_t}$ is a loop if $lh(t) = i$ and $g\restriction [b_t, b_t+2\varepsilon_i]$ is also a loop if $lh(t) = i$ and $0 \leq (c_t)_i < i$.

For $s \in S_m$, let $T_{m,s} = \{t \in T_m : s_t = s\}$. For $t \in T_m$, define t^* so that $t = t^* * (t_{lh(t^*)+1}, \cdots, t_m)$, $(c_t)_{lh(t^*)} < lh(t^*)$, and $(c_t)_i = i$ for $lh(t^*) < i \leq m$. We remark that, $t^* = t$ if and only if $(c_t)_m < m$, and, $t^* = (\ )$ if and only if $(c_t)_i = i$ for $1 \leq i \leq m$.

Since $g\restriction [b_t, b_t+\varepsilon_m]$ is determined only by s_t, if $s_t = s_{t'}$, then $g\restriction [b_t, b_t+\varepsilon_m] \equiv g\restriction [b_{t'}, b_{t'}+\varepsilon_m]$ for $t, t' \in T_m$.

If $t^* = t'^*$ for distinct $t, t' \in T_m$, then $s_t \neq s_{t'}$. Hence the correspondence from t to s_t on $\{t \in T_m : t^* = u\}$ is one to one for $u \in \bigcup_{i=1}^m T_i$ with $u(lh(u)) < lh(u)$ or for $u = (\ )$. In addition, for $u \in \bigcup_{i=1}^m T_i$ with $u(lh(u)) < lh(u)$, we have $g\restriction [b_u+\varepsilon_{lh(u)}, b_u+2\varepsilon_{lh(u)}] \equiv (f\restriction [a_{s_u}, a_{s_u}^+])^-$ and, for $t \in T_m$ with $t^* = (\ )$, we have a corresponding subpath in f^- with which $g\restriction [b_t, b_t+\varepsilon_m]$ forms a loop.

Let $C_m = \{t \in T_m : (c_t)_i = i \text{ for } 1 \leq i \leq m\}$. Since $|\{t \in T_m : s_t = s\}| = (m+1)!$ for $s \in S_m$, we have

$$\begin{aligned} [gf^-]_h \quad = \quad & \Sigma_{i=1}^{m-1}\Sigma_{s\in S_i}(i+1)!\,[l_s]_h \\ & +\Sigma_{s\in S_m}(m+1)!\,\beta_s, \end{aligned}$$

where $\beta_s = [g\restriction [b_t, b_t+\varepsilon_m](f\restriction [a_s, a_s^+])^-]_h$ for $t \in C_m$ with $s_t = s$.

Hence, we have $[gf^-]_h + \alpha_1 - \alpha_m = \Sigma_{s\in S_m}(m+1)!\beta_s$ and $[gf^-]_h + \alpha_1$ is the desired one.

Lemma 4.1. *Let X be a one-dimensional Peano continuum. Then* Ker(σ) *is complete mod-U.*

Proof. Let $\alpha_m \in \mathrm{Ker}(\sigma)$ and $\alpha_m \in \mathrm{Ker}(\sigma)$ and $(m+1)! \mid \alpha_{m+1} - \alpha_m$ in $\mathrm{Ker}(\sigma)$ for $1 \leq m < \omega$. Then we have $\gamma_m \in \mathrm{Ker}(\sigma)$ such that $(m+1)!\,\gamma_m = \alpha_{m+1} - \alpha_m$.

Let $f : [0,1] \to X$ be a path such that $\mathrm{Im}(f) = X$ and $\mathcal{U}_m$ be finite open covers of X such that each $U \in \mathcal{U}_m$ is path-connected, the diameter of each $U \in \mathcal{U}_m$ is less than $1/m$ and $\mathcal{U}_{m+1}$ is a refinement of $\mathcal{U}_m$. To use the preceding construction, we inductively choose k_m in the following way. First k_m should be large so that for each $s \in S$ with $lh(s) = m$ there exits $U \in \mathcal{U}_m$ with $f([a_s, a_{s^+}]) \subseteq U$. By Lemma 3.7 γ_m can be expressed as the sum of homology classes of arbitrary small loops. We want loops in some $U \in \mathcal{U}_m$, hence the number of loops might be large. Second k_m should be large so that γ_m is expressed by k_m loops each of which is in some $U \in \mathcal{U}_m$. Hence we choose k_m which satisfies the two conditions. Since each $U \in \mathcal{U}_m$ is path-connected, a sum of homology classes of loops in U can be replaced by a homologous loop in U. Hence we have $U_s \in \mathcal{U}_{lh(s)}$ and loops l_s in U_s with base point $f(a_s)$ so that

$$\gamma_m = \Sigma_{lh(s)=m}[l_s]_h.$$

Then we have $\alpha_{m+1} = \Sigma_{i=1}^m \Sigma_{s\in S_i}\,(i+1)!\,[l_s]_h + \alpha_1$ in $H_1(X)$ for $m \geq 1$. Now, the assumptions for the preceding construction are satisfied and we have the desired element $[gf^-]_h + \alpha_1$. □

Lemma 4.2. [11, Theorem 2.1] *Let X be a one-dimensional normal space. Then $H_1(X)$ is torsionfree.*

Now, according to the facts in Section 2 Lemmas 4.1 and 4.2 imply

Lemma 4.3. *Let X be a one-dimensional Peano continuum. Then* $\mathrm{Ker}(\sigma)$ *is algebraically compact.*

Lemma 4.4. *(Folklore) Let X be a one-dimensional Peano continuum. If X is semi-locally simply connected, then the Čech homology group $\check{H}_1(X)$ is isomorphic to a free abelian group of finite rank. Otherwise, $\check{H}_1(X)$ is isomorphic to $\mathbb{Z}^\omega$.*

Next we construct loops whose homotopy classes are in $Ker(\sigma)$ and the homology classes which generate pure subgroups of $H_1(X)$ when X is not locally semi-simply connected. Suppose that X is not locally simply connected at $x_0 \in X$.

First lemma is well-known and it can be proved using arbitrarily small simple closed curves and we omit its proof.

Lemma 4.5. *Let X be a one-dimensional space Peano continuum which is not semi-locally simply-connected at x_0. Then there exists a closed subspace Y such that (Y, x_0) is homotopy equivalent to the Hawaiian earring* $(\mathbb{H}, o)$.

Then we have a dendrite D in Y such that $Y \setminus D$ consists of countable open arcs A_n which converge to x_0 by [7, Theorem 1.2] with its proof.

We construct certain reduced loops in Y. Let l_n be a reduced loop which starts from x_0, reach a one end of A_n in D, goes through A_n and goes back to x_0 in D. We call this direction of A_n to be plus and the reverse direction to be minus.

Let l^*_n be the reduced loop of $l_{2n}l_{2n+1}l^-_{2n}l^-_{2n+1}$, i.e l^*_n goes plus A_{2n}, plus A_{2n+1}, minus A_{2n} and minus A_{2n+1} when we disregard D. We call this last property $(*_n)$ for simplicity. Moreover, the reduced loops of $l^*_0 \cdots l^*_m$ for $m \geq n$ also has this property $(*_n)$. Let l^* be the reduced loop of the infinite concatenation $l^*_0 \cdots l^*_n \cdots$. Then we see that, for each $\delta > 0$, $l^* \restriction [1-\delta, 1]$ has the property $(*_n)$ for sufficiently large n and for each n there exists $\delta > 0$ such that $l^* \restriction [0, 1-\delta]$ has the property $(*_n)$. We remark that l^{*-} has not the property $(*_n)$.

For a non-degenerate path $f : [0,1] \to X$, a *tail* of f is a subpath $f \restriction [1-\delta, 1]$ for some $\delta > 0$. The following lemma is straightforward and we omit its proof.

Lemma 4.6. *Let $f_0 \cdots f_k$ be a reduced path. There exists a tail m_0 of l^* such that every subpath m in $f_0 \cdots f_k$ which is equivalent to m_0 or m_0^- is a subpath of some f_i.*

Lemma 4.7. *The homology class $[l^*]_h$ generates a pure subgroup of $H_1(X)$ which is isomorphic to $\mathbb{Z}$.*

Proof. Since $H_1(X)$ is torsionfree, it is sufficient to show that $[l^*]_h$ is not divided by any $n \geq 2$. To show by contradiction, suppose that $[l^*]_h$ is divided by some $n \geq 2$. Then we have a cyclically reduced loop l and a reduced path such that plp^- is a reduced with base point x_0 and $l^*pl^np^-$ is of 0-form among paths in X. We argue dividing to cases.

(Case 1) p is degenerate and l^*l^n is reduced:

We have $l^*l^n \equiv f_1 \cdots f_k$ where $f_1, \cdots, f_k$ are paired forming 0-form. By Lemma 4.6 we have a tail m_0 which satisfies the property in the lemma for $l^*l \cdots l$ and $f_1 \cdots f_k$ under these presentations. Then the number of occurrences of m_0 is the same as that of m_0^- in $f_1 \cdots f_k$. Let a^+ be the number of occurrences of m_0 in l and a^- be the number of occurrences of m_0 in l. Then we have $na^+ + 1 = na^-$ and hence $n(a^- - a^+) = 1$, which contradicts $n \geq 2$.

(Case 2) p is non-degenerate and $l^*pl^np^-$ is reduced:

We choose m_0 similarly to Case 1 considering p and p^-. Since the number of occurrences of m_0 in p is the same as that of m_0^- in p^- and that of m_0^- in p is the same as that of m_0 in p^-, we have a contradiction as in Case 1.

(Case 3) p is degenerate and l^*l^n is not reduced:

Since there is a tail t of l^* such that t^- is a a head of l, the reduced loop of l^*l^n of the form $q_0q_2l^{n-1}$ where $q_0q_1 \equiv l^*$ and $q_1q_2 \equiv l$. Using the presentation $q_0q_2q_1 \cdots q_1q_2$ and the 0-form, we choose m_0. Let a^+ be the number of occurrences of m_0 in $l \equiv q_1 q_2$ and a^- be the number of occurrences of m_0 in l as before. Since m_0^- occurs once in q_1 and m_0 does not, we have $n - 1 + n(a^+ - 1) = na^-$ and hence $n(a^+ - a^-) = 1$, which is a contradiction.

(Case 4) p is non-degenerate and $l^* p l^n p^-$ is not reduced: For a sufficiently small tail m_0 of l^*, we have $q_0 m_0 \equiv l^*$ and $m_0^- p_0 \equiv p$. Then in the reduction of $q_0 p_0 l^n p_0^- m_0$ any tail of l or its inverse is canceled. Hence we have a contradiction as in (Case 2). □

Lemma 4.8. *Let X be a one-dimensional normal space. If Y is a path-connected subspace of X, then $H_1(Y)$ is a subgroup of $H_1(X)$.*

Proof. Since every element of $H_1(Y)$ is a homology class of a loop in Y, we let l to be a reduced loop in Y. We only deal with the case that l is non-degenerate. Since the reduced loop of a loop is in the image of the original loop, the reducedness does not depend on whether we consider in X or in Y. Suppose that the homotopy class of l belongs to a commutator subgroup of $\pi_1(X)$. Then l is equivalent to a 0-form where each paths are generally in X, but Lemma 3.6 implies that each path is in Y. Therefore, $H_1(Y)$ is a subgroup of $H_1(X)$. □

Proof of Theorem 1.1. Let $h : H_1(X) \to \mathbb{Z}$ be a homomorphism. By lemma 4.1 we have $h(\mathrm{Ker}(\sigma)) = \{0\}$ and consequently by Lemma 3.8 we have $\mathrm{Ker}(\sigma) = R_{\mathbb{Z}}(H_1(X))$. Therefore $H_1(X)/\mathrm{Ker}(\sigma)$ is a subgroup of the direct product of copies of $\mathbb{Z}$, which is obviously torsionfree. By Lemma 4.3 this implies that $\mathrm{Ker}(\sigma)$ is a direct summand. If X is semi-locally simply-connected, then it is well-known that $H_1(X)$ is a free abelian group of finite rank. Otherwise, we have $\check{H}_1(X) \cong \mathbb{Z}^\omega$ and hence $H_1(X) \cong \mathrm{Ker}(\sigma) \oplus \mathbb{Z}^\omega$. Since there exists a subspace of X which is homotopy equivalent to the Hawaiian earring $\mathbb{H}$, the divisible part $D(H_1(X))$ contains $D(H_1(\mathbb{H})) \cong \oplus_{\mathbf{c}} \mathbb{Q}$ by Lemma 4.8. Since the cardinality of $H_1(X)$ is equal to or less than $\mathbf{c}$, we have $D(H_1(X)) \cong \oplus_{\mathbf{c}} \mathbb{Q}$. The remaining task is to determine the cardinality about reduced algebraically compact group.

Since $\sigma([l^*]_h) = 0$ for l^* in Lemma 4.7, we see $[l^*]_h$ generates a pure subgroup of $Ker(\sigma)$. To show that $Ker(\sigma)$ contains a pure subgroup isomorphic to a free abelian group of the continuum rank we modify the construction of l^* as in the proof of [11, Lemma 3.5]. There exists an almost disjoint family consisting of infinite sets of integers, where S and T is almost disjoint if $S \cap T$ is finite. Let l_S^* be the reduced loop of $l_{i_0}^* \cdots l_{i_n}^* \cdots$, where $i_0 < \cdots < i_n < \cdots$ is the enumeration of S in the order of the integers.

Now it suffices to show that $l^*_{S_1}, \cdots, l^*_{S_n}$ is linearly independent for an almost disjoint family $S_1, \cdots, S_n$. We have a finite set F of integers such that $S_i \cap S_j \subseteq F$ for distinct i, j. For a set S of integers let $r_S : Y \to Y$ be a retraction such that $r_S(A_n) \subseteq D$ for $n \notin S$ and $r_S \restriction A_n$ is the identity for $n \in S$. Let $\lambda_1[l^*_{S_1}]_h + \cdots + \lambda_n[l^*_{S_n}]_h = 0$. By Lemma4.8, we may work in Y. Let $S = S_i \setminus F$. Since $(r_S)_*([l^*_{S_j})$ is trivial for $j \neq i$ but $S \neq \emptyset$ and $H_1(X)$ is torsionfree, $(r_S)_*([l^*_{S_i}]_h)$ is non-zero and hence $\lambda_i = 0$. □

Remark 4.9. Here we show that the compactness of a space is essential for the algebraical compactness of $\mathrm{Ker}(\sigma)$ in Lemma 4.3. Let X be a subspace of the plane obtained by attaching copies of $\mathbb{H}$ on the half line $\{0\} \times [0, \infty)$, i.e.

$$\begin{aligned} X &= \{0\} \times [1, \infty) \cup \\ &\quad \{(x, y) : (x - \frac{1}{n})^2 + (y - m)^2 = \frac{1}{n^2} : 3 \leq n < \omega, 1 \leq m < \omega\}. \end{aligned}$$

Then X is locally path-connected, path-connected, separable metric space. In the m-th copy of the Hawaiian earring, we have a non-trivial element α_m in $\mathrm{Ker}(\sigma)$ such that $\langle [\alpha_m]_h \rangle$ is a pure subgroup of $H_1(X)$, where σ is the natural homomorphism to the Čech homology group. Then we have

$$(m+1)! \mid \Sigma_{i=1}^{m+1} i\,! \, [\alpha_i]_h - \Sigma_{i=1}^{m} i\,! \, [\alpha_i]_h.$$

Suppose that $\mathrm{Ker}(\sigma)$ is algebraically compact. Then we have a loop l such that $(m+1)! \mid [\, l\,]_h - \Sigma_{i=1}^{m} i\,! \, [\alpha_i]_h$ for each $1 \leq m < \omega$. Since the image of l is compact, we have m_0 such that

$$\begin{aligned} \mathrm{Im}(l) &\subseteq \{0\} \times [1, m_0 - 1] \cup \\ &\quad \{(x, y) : (x - \frac{1}{n})^2 + (y - m)^2 = \frac{1}{n^2} : 3 \leq n < \omega, 1 \leq m \leq m_0 - 1\}. \end{aligned}$$

Considering the retraction of X to

$$\{(x, y) : (x - \frac{1}{n})^2 + (y - m_0)^2 = \frac{1}{n^2} : 3 \leq n < \omega\},$$

we conclude $(m_0 + 1)! \mid -m_0![\alpha_{m_0}]_h$. Since $H_1(X)$ is torsionfree, we have $m_0 + 1 \mid [\alpha_{m_0}]_h$, which contradicts that $\langle [\alpha_{m_0}]_h \rangle$ is a pure subgroup.

Though $\mathrm{Ker}(\sigma)$ may not be algebraically compact for a non-compact space X, we have the following.

Theorem 4.10. *Let X be a one-dimensional locally path-connected metric space. Then* $\mathrm{Ker}(\sigma) = R_{\mathbb{Z}}(H_1(X))$.

Proof. By lemma 3.8 it suffices to show that $\mathrm{Ker}(\sigma) \leq R_{\mathbb{Z}}(H_1(X))$. Since each path-connected component is open by the local path-connectivity, the

Čech homology group is the direct product of the Čech homology groups of path-connected components. Hence without loss of generality we may assume that X is path-connected. Let l be a loop with $[\,l\,]_h \in \mathrm{Ker}(\sigma)$ and $h : H_1(X) \to \mathbb{Z}$ be a homomorphism. We define a map $\varphi : \widehat{\mathbb{Z}} \to \mathrm{Ker}(\sigma)$ such that $h \circ \varphi$ becomes to be a homomorphism. For $u \in \widehat{\mathbb{Z}}$, i.e. $u = \Sigma_{i=1}^{\infty} m!\, a_m$ where $0 \leq a_m \leq m$, we define a loop l_u as follows. We modify the construction in the proof of Lemma 4.1. Replace f by l and for each a_m we express $a_m[\,l\,]_h$ as the sum of homology classes of loops each of which is in some $U \in \mathcal{U}_m$. Then we have a loop l_u such that

$$(m+1)! \mid [l_u]_h - \Sigma_{i=1}^{m} i!\; a_i[\,l\,]_h.$$

Let $\varphi(u) = [l_u]_h$. For $u, v \in \widehat{\mathbb{Z}}$, let $u = \Sigma_{i=1}^{\infty} i!\, a_i$, $v = \Sigma_{i=1}^{\infty} i!\, b_i$ and $u + v = \Sigma_{i=1}^{\infty} i!\, c_i$ where $0 \leq a_i, b_i, c_i \leq i$. Since

$$(m+1)! \mid \Sigma_{i=1}^{m} i!\; c_i - (\Sigma_{i=1}^{m} i!\; a_i + \Sigma_{i=1}^{m} i!\; b_i),$$

we have

$$(m+1)! \mid h([l_{u+v}]_h) - (h([l_u]_h + h([l_v]_h))$$

for every m and hence $h \circ \varphi(u + v) = h \circ \varphi(u) + h \circ \varphi(v)$. Since $\mathbb{Z}$ is cotorsionfree, $h \circ \varphi$ is a trivial homomorphism, which implies $h([\,l\,]_h) = h \circ \varphi(1) = 0$. □

Remark 4.11. According to Theorem 1.1 $H_1(X)/R_{\mathbb{Z}}(H_1(X))$ is isomorphic to a free abelian group of finite rank or $\mathbb{Z}^{\omega}$. Even for one-dimensional locally path-connected separable metric spaces X, $H_1(X)/R_{\mathbb{Z}}(H_1(X))$ are abundant. For this we refer the reader to [13, Section 6], we defined a factor $H_n^T(X)$ of the singular homology group $H_n(X)$ and in our case $H_1^T(X) \cong H_1(X)/R_{\mathbb{Z}}(H_1(X))$ holds. There we see the abundance of $H_1^T(X)$. The spaces defined there are not metrizable, but by a standard method inducing metrizable topology we have metrizable spaces X with the same $H_1(X)$ and $H_1^T(X)$.

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