Group theoretic properties for wild algebraic topology

Katsuya Eda

Department of Mathematics, Waseda University, Tokyo 169-8555, JAPAN

2011 July

Specker's Theorem and ...

Theorem 1. (E. Specker (1950))
$\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) = \langle p_n : n < \omega \rangle$. Consequently,
$\mathrm{Hom}(\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}^\omega$ canonically.

Notice that the cardinality of $\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Q})$ is $2^{2^{\aleph_0}}$!
Here is a world of Duality.

J. Łoś (1955) introduced a notion of slender abelian groups S, i.e. $\mathrm{Hom}(\mathbb{Z}^\omega, S) = \oplus_{n<\omega} \mathrm{Hom}(\mathbb{Z}_n, S)$.
R. J. Nunke (1962) characterized the slenderness:
"An abelian group A is slender, if and only if A is torsionfree and contains neither $\mathbb{Z}^\omega$, nor the divisible group $\mathbb{Q}$, nor the algebraically compact group $\mathbb{J}_p$ for any prime p."

(continued)

S.U. Chase (1962) investigated homomorphisms from direct products to direct sums of modules. For torsionfree abelian groups A_n and B_i it is stated as:
Let $h : \Pi_{n<\omega} A_n \longrightarrow \oplus_{i \in I} B_i$. Then there exist n_0 and a finite subset F of I such that

$$h(\Pi_{n \geq n_0} A_n) \subseteq \oplus_{i \in F} B_i + D(\oplus_{i \in I} B_i)$$

where $D(\oplus_{i \in I} B_i)$ is the divisible subgroup.

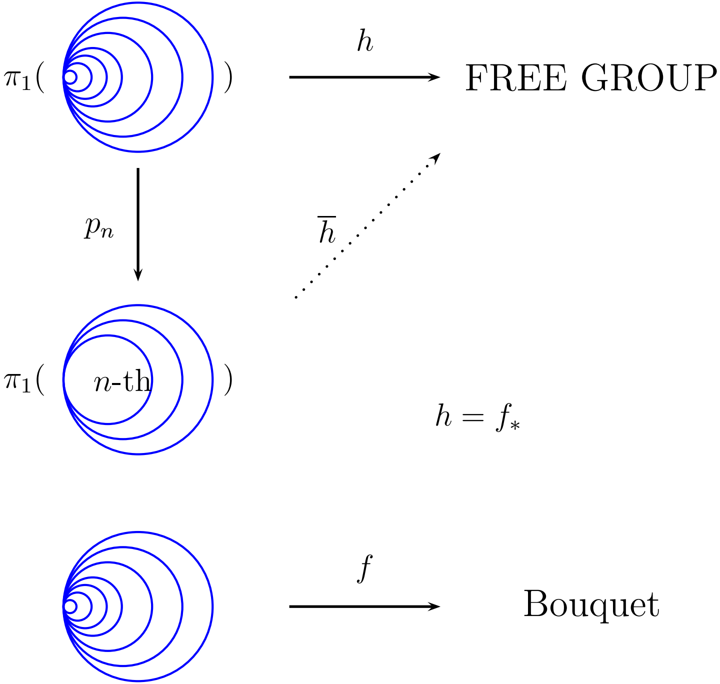
A large part is mapped into a small part!

All of these are called "Specker phenomenon".

Noncommutative versions

Higman's Theorem (1952) in a topological form:
Any homomorphism from the fundamental group or the shape group of the Hawaiian earring $\mathbb{H}$ to free groups factors through the fundamental group of a finite bouquet.

In comparison with Specker's theorem Higman's theorem has not attracted attentions for a long time. But now Higman's theorem, the non-commutative Specker phenomenon, is the central concept in Wild Algebraic Topology, a world of non-commutative Duality.



$\pi_1($ $)$
h
FREE GROUP
p_n
$\overline{h}$
$\pi_1($ n-th $)$
$h = f_*$
f
Bouquet

Higman's theorem for shape groups

Let $\check{\pi}_1(X)$ be the shape (Čech homotopy) group of X. Higman's theorem was strengthened.
Theorem 2. ([E2] 1998) Every homomorphism from the shape group $\check{\pi}_1(\mathbb{H})$ to the fundamental group $\pi_1(\mathbb{H})$ factors through the fundamental group of a finite bouquet.

Non-commutatively slender groups

Abelian groups are n-slender, if and only if they are slender. Basic properties are in [E3](1992). Recently J. Nakamura proved that the surfaces groups except a torsion case are n-slender. It is unknown whether every finitely generated torsion-free groups are n-slender or not.

Noncommutative Chase's lemma

Theorem 3. ([E1] 2011) Let G_i $(i \in I)$ and H_j $(j \in J)$ be groups and $h : \times^{\sigma}_{i \in I} G_i \to *_{j \in J} H_j$ be a homomorphism from the free σ-product of groups G_i to the free product of groups H_j. Then there exist a finite subset F of I and $j \in J$ such that $h(\times^{\sigma}_{i \in I \setminus F} G_i)$ is contained in a subgroup which is conjugate to H_j.

In a version of [E3] (1992) the conclusion was: $h(\times^{\sigma}_{i \in I \setminus F} G_i)$ is contained in $*_{j \in J_0} H_j$ for some finite $J_0 \subseteq J$.

A large part is mapped into a small part!

In case the range is an n-slender group, a small part is the trivial group.

Complementary parts of the Specker phenomenon

The Specker phenomenon occurs when the target groups are thin or sparced, mathematically speaking, slender groups, n-slender groups, direct sums, or free products. There are complementary notions.

Let $\sigma : \pi_1(\mathbb{H}) \to \mathbb{Z}^\omega$. Then $\mathrm{Ker}(\sigma)$ is complete mod-U and contains the divisible subgroup $\oplus_{2^{\aleph_0}} \mathbb{Q}$ ([E3] 1992).
An abelian group A is complete mod-U, if for a given sequence $a_n (n < \omega)$ of elements of A satisfying $n! \,|\, a_{n+1} - a_n$ for every $n < \omega$ there exists a_∞ such that $n! \,|\, a_\infty - a_n$ for every $n < \omega$.
Equivalently $A/U(A)$ is comlete under the topology induced from $\{n!A/U(A) \,:\, n < \omega\}$, where $U(A) = \bigcap_{n<\omega} n!A$.

(continued)

Abelian group A is algebraically compact, equivalently pure-injective, if and only if A is complete mod-U and $UU(A) = U(A)$.

Torsionfree algebraically compact groups have simple structures consisting of the divisible group $\mathbb{Q}$ and the p-adic integer group $\mathbb{J}_p$.

Compare with Nunke's characterization of slender abelian groups.

Note that $\mathrm{Hom}(A, \mathbb{Z}) = \{0\}$ for A complete mod-U, more generally $\mathrm{Hom}(A, B) = \{0\}$ for cotorsionfree abelian group B.

These complementary parts concerns the abelianization, i.e. singular homology.

Wild points are related to RED properties.

(continued)

Theorem 4. ([E4] 1991) If spaces X and Y are first countable at $x \in X$ and $y \in Y$ respectively, then $H_1((CX, x) \vee (CY, y))$ is complete mod-U.
In general
$H_1((CX, x) \vee (CY, y))/U(H_1((CX, x) \vee (CY, y)))$ is a union of complete subgroups.
Consequently, $H^1((CX, x) \vee (CY, y))$ is trivial and $H^1(X \vee Y) = H^1(X) \oplus H^1(Y)$.
Theorem 5. ([EK] 2000) The integral singular homology group $\mathrm{H}_1(\mathbb{H})$ is isomorphic to

$$\mathbb{Z}^{\omega} \oplus \Pi_{p:\text{prime}} A_p \oplus \oplus_{2^{\aleph_0}} \mathbb{Q},$$

where A_p is isomorphic to the p-adic completion of $\oplus_{2^{\aleph_0}} \mathbb{J}_p$.

New Properties

A functor

$$U_\infty(A) = \langle a \in A \ : \ n|a \text{ for infinitely many } n < \omega \rangle$$

was defined in Appendix of Cannon-Conner [CC]. This is neither a radical nor a socle and hence its group theoretic meaning is some what difficult to understand, where a radical R satisfies $R(A/R(A)) = \{0\}$ and a socle S does $S(S(A)) = S(A)$.

We have $U_\infty(\mathbb{J}_p) = \mathbb{J}_p$, while $U(\mathbb{J}_p) = \{0\}$. If $U_\infty(A) = A$, then $\mathrm{Hom}(A, \mathbb{Z}) = \{0\}$. Not much is known about this functor.
There are descriptions on abelian groups in Appendix of Cannon-Conner [CC].

(continued)

H. Fischer proposed the following property of a group G.

> **For every Peano continuum X and every homomorphism $h : \pi_1(X) \to G$,**

$$\bigcap\{h(\pi_1(\mathcal{U})) \ : \mathcal{U} \text{ covering on } X\} = \{e\}.$$

If G is n-slender, G has this property. Actually, G is n-slender if and only if for every Peano continuum X and any homomorphism $h : \pi_1(X) \to G$ there exists a covering $\mathcal{U}$ on X satisfying $h(\pi_1(\mathcal{U})) = \{e\}$ if and only if for any homomorphism $h : \pi_1(\mathbb{H}) \to G$ there exists a covering $\mathcal{U}$ on $\mathbb{H}$ satisfying $h(\pi_1(\mathcal{U})) = \{e\}$.

If we restrict to the case that G is abelian, $\bigcap\{h(\pi_1(\mathcal{U})) \ : \mathcal{U} \text{ covering on } \mathbb{H}\} = \{0\}$ holds if and only if G is cotorsionfree.

References

[CC] J. W. Cannon and G. R. Conner, The combinatorial structure of the Hawaiian earring group, Topology Appl. 106 (2000), 225–271.

[Ch] S. U. Chase, On direct sums and products of modules, Pacific J. Math. 12 (1962) 847–854.

[E1] K. Eda, Atomic property of the fundamental groups of the Hawaiian earring and wild locally path-connected spaces, J. Math. Soc. Japan, 63 (2011), 769–787.

[E2] K. Eda, The non-commutative Specker phenomenon, J. Algebra, 204 (1998), 95–107.

[E3] K. Eda, Free σ-products and noncommutatively slender groups, J. Algebra 148 (1992), 243–263.

References (continued)

[E4] K. Eda, The first integral singular homology groups of one point unions, Quart. J. Math. 42 (1991), 443–456.

[EK] K. Eda and K. Kawamura, The singular homology of the Hawaiian earring, J. London Math. Soc., 62 (2000), 305–310.

[H] G. Higman, Unrestricted free products, and variety of topological groups, J. London Math. Soc. 27 (1952), 73–81.

[N] R. J. Nunke, Slender groups, Acta Sci. Math. (Szeged) 23 (1962), 67–73.

[S] E. Specker, Additive gruppen von folgen ganzer zahlen, Portugal. Math. 9 (1950), 131–140.